# CSE 417
# Algorithms and Complexity

Lecture 23

Winter 2020

Network Flow, Part 2

## Outline

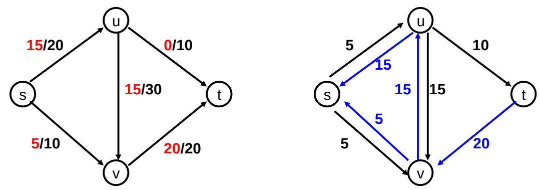
- ~~Network flow definitions~~
- ~~Flow examples~~
- ~~Augmenting Paths~~
- ~~Residual Graph~~
- Ford Fulkerson Algorithm
- Cuts
- Maxflow-MinCut Theorem
- Simple applications of Max Flow

## Network Flow Definitions

- Flowgraph: Directed graph with distinguished vertices s (source) and t (sink)
- Capacities on the edges, c(e) >= 0
- Problem, assign flows f(e) to the edges such that:
  - 0 <= f(e) <= c(e)
  - Flow is conserved at vertices other than s and t
    - Flow conservation: flow going into a vertex equals the flow going out
  - The flow leaving the source is a large as possible

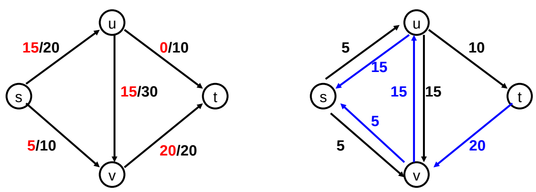
## Residual Graph

- Flow graph showing the remaining capacity
- Flow graph G, Residual Graph $G_R$
  - G: edge e from u to v with capacity c and flow f
  - $G_R$: edge e' from u to v with capacity c – f
  - $G_R$: edge e'' from v to u with capacity f




## Augmenting Path Algorithm

- Augmenting path in residual graph
  - Vertices $v_1, v_2, \ldots, v_k$
    - $v_1 = s$, $v_k = t$
    - Possible to add b units of flow between $v_j$ and $v_{j+1}$ for j = 1 … k-1




## Adding flow along a path in the residual graph

- Let P be an s-t path in the residual graph with capacity b
- b units of flow can be added along P in the graph G
- Need to show:
  - new flow satisfies capacity constraints
  - new flow satisfies conservation constraints

## Ford-Fulkerson Algorithm (1956)

while not done

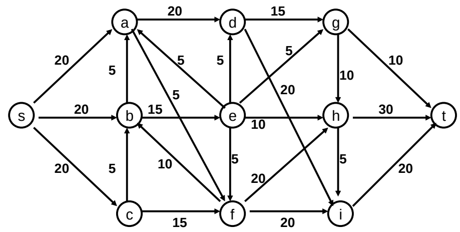
Construct residual graph $G_R$

Find an s-t path P in $G_R$ with capacity b > 0

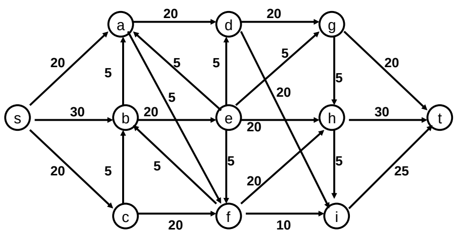
Add b units of flow along path P in G

If the sum of the capacities of edges leaving S is at most C, then the algorithm takes at most C iterations

## Flow Example I




## Flow Example II



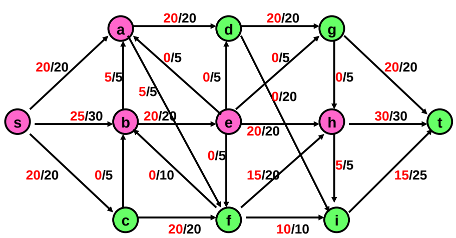

## Cuts in a graph

- Cut: Partition of V into disjoint sets S, T with s in S and t in T.
- Cap(S,T): sum of the capacities of edges from S to T
- Flow(S,T): net flow out of S
  - Sum of flows out of S minus sum of flows into S

- Flow(S,T) <= Cap(S,T)

## What is Cap(S,T) and Flow(S,T)

S={s, a, b, e, h}, T = {c, f, i, d, g, t}

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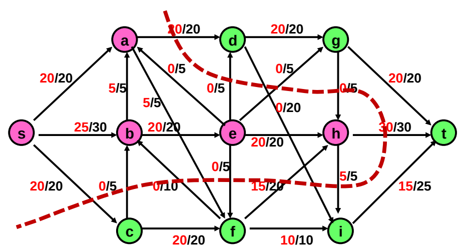
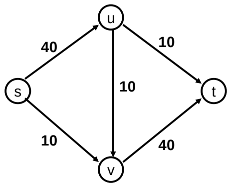
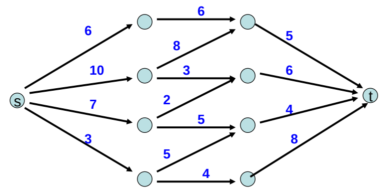



Cap(S,T) = 95, Flow(S,T) = 80 – 15 = 65

# Minimum value cut




# Find a minimum value cut

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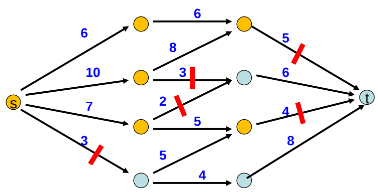


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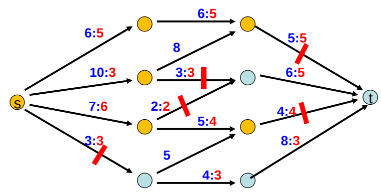
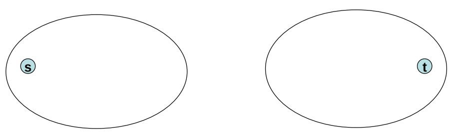



# MaxFlow – MinCut Theorem

- There exists a flow which has the same value of the minimum cut
- Proof: Consider a flow where the residual graph has no s-t path with positive capacity
- Let S be the set of vertices in $G_R$ reachable from s with paths of positive capacity

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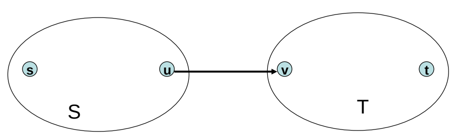


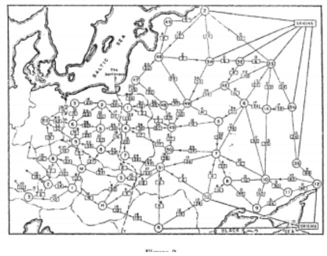

What can we say about the flows and capacity between u and v?

## Max Flow - Min Cut Theorem

- Ford-Fulkerson algorithm finds a flow where the residual graph is disconnected, hence FF finds a maximum flow.
- If we want to find a minimum cut, we begin by looking for a maximum flow.

## History

- Ford / Fulkerson studied network flow in the context of the Soviet Rail Network

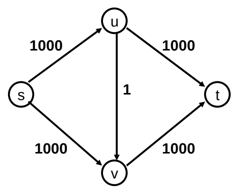


## Ford Fulkerson Runtime

- Cost per phase X number of phases
- Phases
  - Capacity leaving source: C
  - Add at least one unit per phase
- Cost per phase
  - Build residual graph: O(m)
  - Find s-t path in residual: O(m)

## Performance

- The worst case performance of the Ford-Fulkerson algorithm is horrible




## Better methods of finding augmenting paths

- Find the maximum capacity augmenting path
  - $O(m^2 \log(C))$ time algorithm for network flow
- Find the shortest augmenting path
  - $O(m^2 n)$ time algorithm for network flow
- Find a blocking flow in the residual graph
  - $O(mn \log n)$ time algorithm for network flow