CSE 417 Algorithms

Lecture 17, Winter 2020

Divide and Conquer

Dynamic Programming

Announcements

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Divide and Conquer Algorithms

- Mergesort, Quicksort
- Strassen's Algorithm
- Median
- Inversion counting
- Closest Pair Algorithm (2d)
- Integer Multiplication (Karatsuba's Algorithm)

Integer Arithmetic

```
    9715480283945084383094856701043643845790217965702956767
+   1242431098234099057329075097179898430928779579277597977
```

Runtime for standard algorithm to add two n digit numbers:

```
    2095067093034680994318596846868779409766717133476767930
X   5920175091777634709677679342929097012308956679993010921
```

Runtime for standard algorithm to multiply two n digit numbers:

Recursive Multiplication Algorithm (First attempt)

$x = x_1 2^{n/2} + x_0$

$y = y_1 2^{n/2} + y_0$

$xy = (x_1 2^{n/2} + x_0) (y_1 2^{n/2} + y_0)$

$= x_1 y_1 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0$

Recurrence:

Run time:

Simple algebra

$x = x_1 2^{n/2} + x_0$

$y = y_1 2^{n/2} + y_0$

$xy = x_1 y_1 2^n + (x_1 y_0 + x_0 y_1) 2^{n/2} + x_0 y_0$

$p = (x_1 + x_0)(y_1 + y_0) = x_1 y_1 + x_1 y_0 + x_0 y_1 + x_0 y_0$

Karatsuba's Algorithm

Multiply n-digit integers x and y

Let $x = x_1 2^{n/2} + x_0$ and $y = y_1 2^{n/2} + y_0$

Recursively compute

$a = x_1 y_1$

$b = x_0 y_0$

$p = (x_1 + x_0)(y_1 + y_0)$

Return $a2^n + (p - a - b)2^{n/2} + b$

Recurrence: $T(n) = 3T(n/2) + cn$

$\log_2 3 = 1.58496250073\ldots$

Fast Integer Multiplication

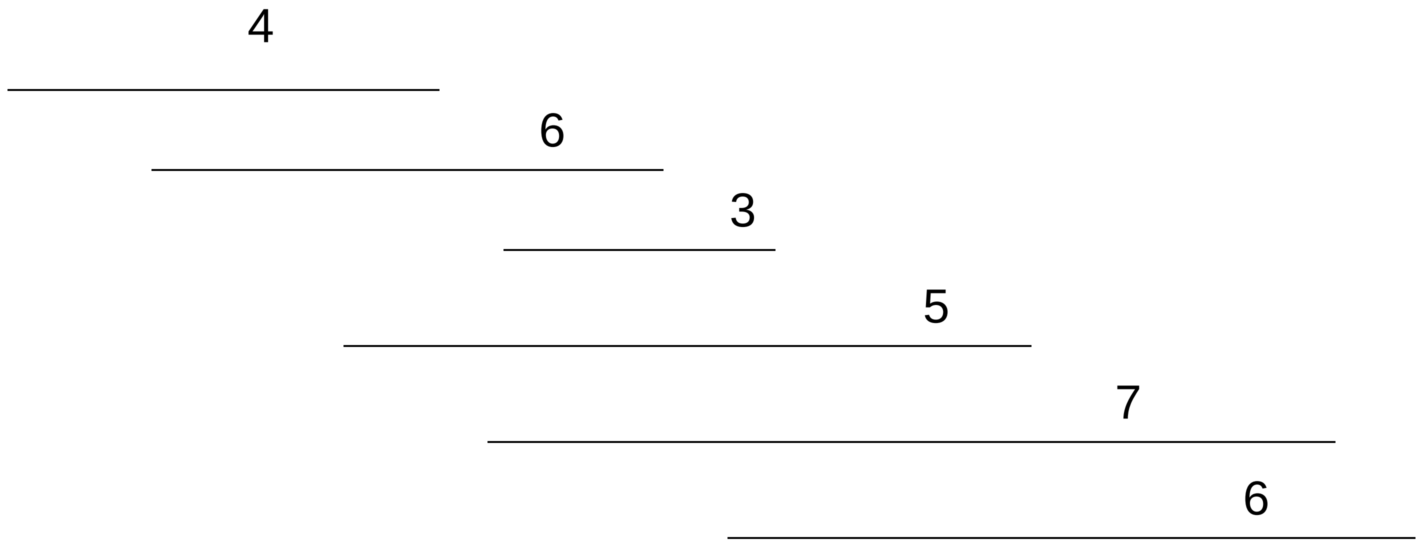
- Grade School $O(n^2)$
- Karatsuba $O(n^{1.58})$
- Toom-Cook $O(n^{1.46})$ [For 3 pieces]
  - $O(n^{1+eps})$ [For k pieces]
- Schonhage-Strassen
  - Fast Fourier Transform based algorithm
  - $O(n \log n \ \log\log n)$
  - Becomes practical for ~25,000 digits

Dynamic Programming

Intervals sorted by end time

Dynamic Programming

- Weighted Interval Scheduling
- Given a collection of intervals $I_1,\ldots,I_n$ with weights $w_1,\ldots,w_n$, choose a maximum weight set of non-overlapping intervals



Intervals sorted by end time

Optimality Condition

- Opt[j] is the maximum weight independent set of intervals I_1, I_2, . . ., I_j
- Opt[j] = max(Opt[j – 1], w_j + Opt[p[j]])
  - Where p[j] is the index of the last interval which finishes before I_j starts

Algorithm

```
MaxValue(j) =
   if j = 0 return 0
   else
        return max( MaxValue(j-1),
                    w_j + MaxValue(p[ j ]))
```

Worst case run time: 2^n

A better algorithm

M[j] initialized to -1 before the first recursive call for all j

```
MaxValue(j) =
    if j = 0 return 0;
    else if M[ j ] != -1 return M[ j ];
    else
        M[ j ] = max(MaxValue(j-1), w_j + MaxValue(p[ j ]));
        return M[ j ];
```

Iterative Algorithm

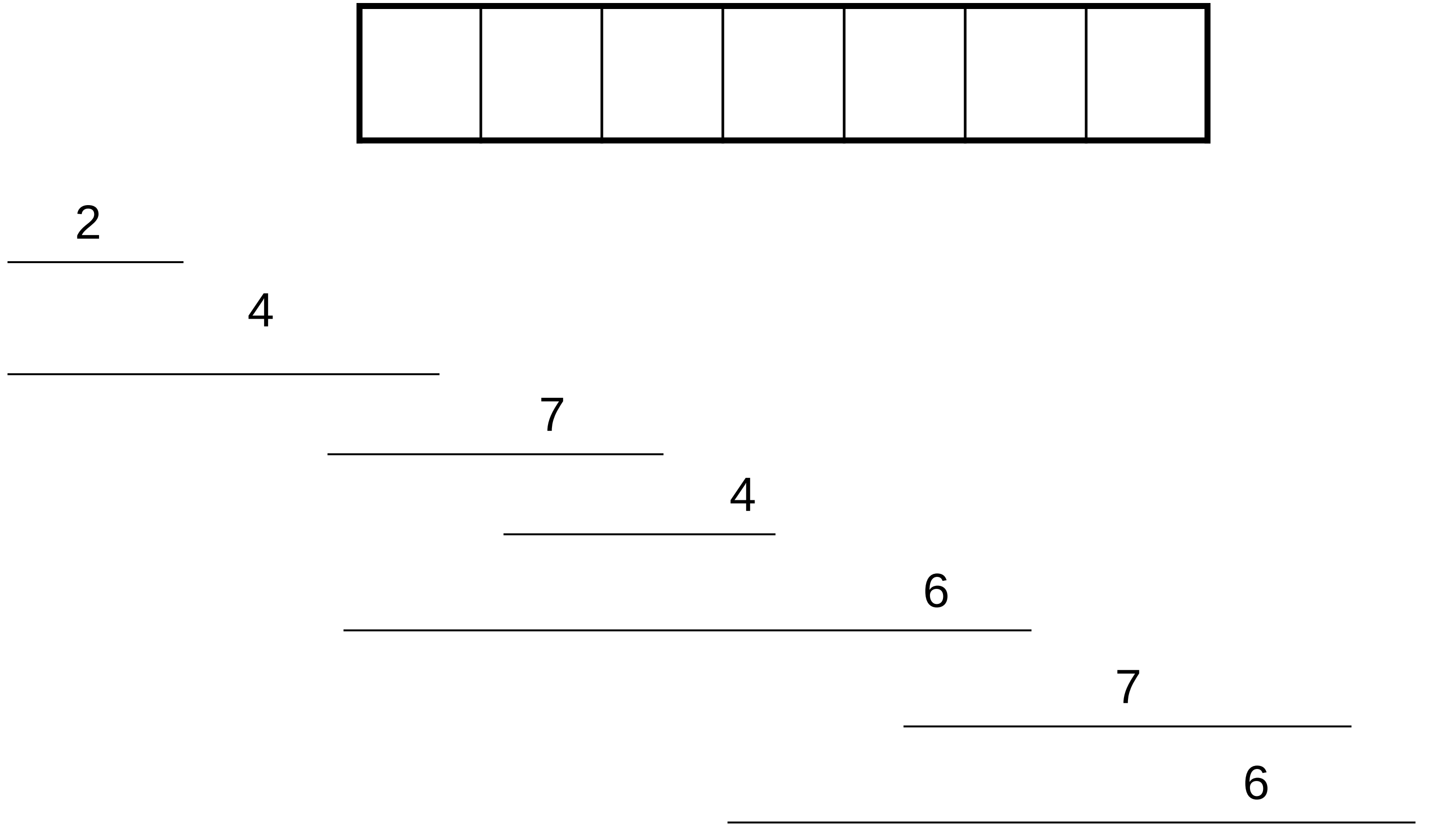
Express the MaxValue algorithm as an iterative algorithm

```
MaxValue {

}
```

Fill in the array with the Opt values

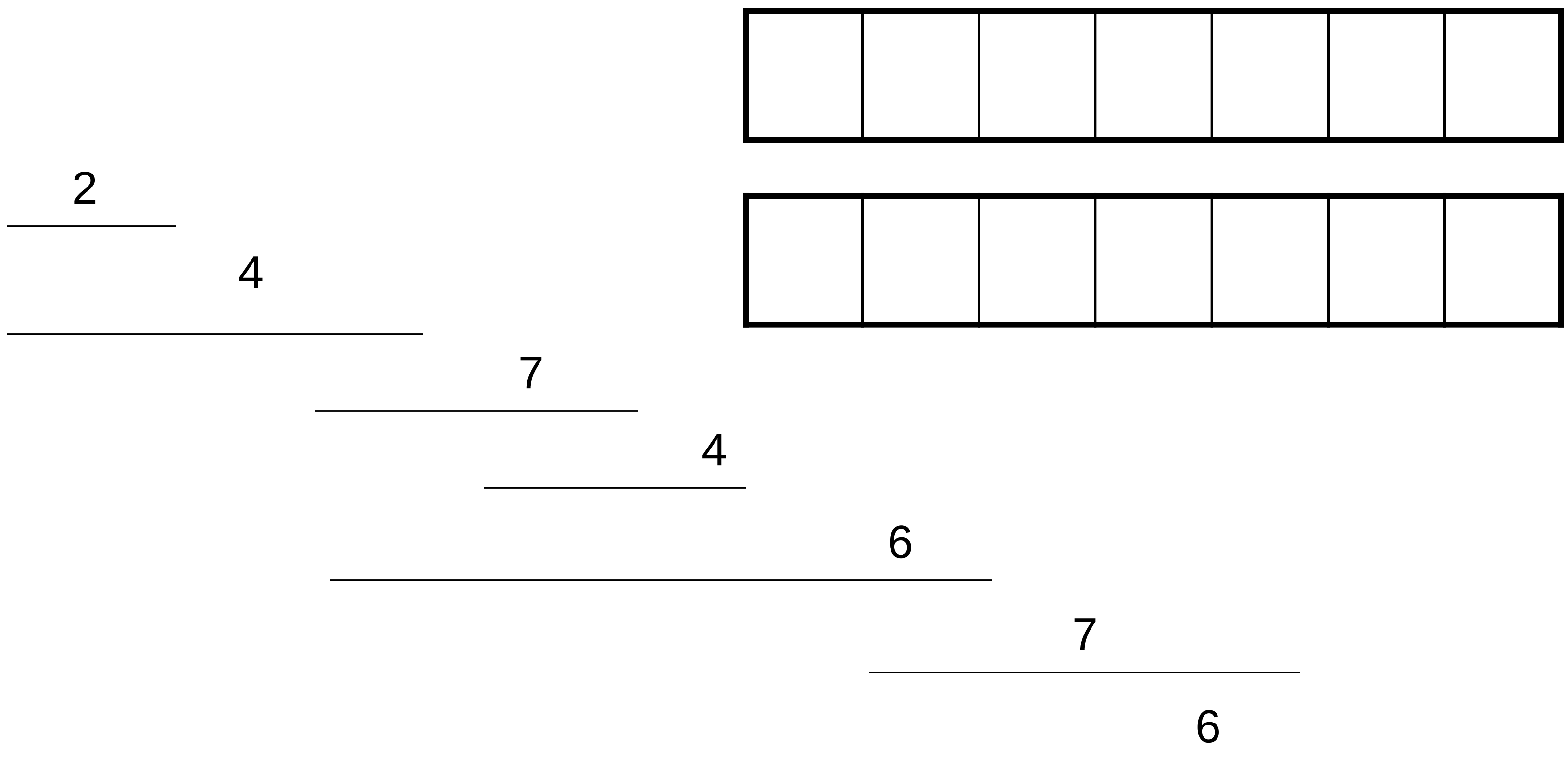
$$Opt[\, j\,] = \max\,(Opt[\, j - 1],\ w_j + Opt[\, p[\, j\,]\,])$$



Computing the solution

Opt[j] = max (Opt[j – 1], w_j + Opt[p[j]])

Record which case is used in Opt computation



Dynamic Programming

- The most important algorithmic technique covered in CSE 421
- Key ideas
  - Express solution in terms of a polynomial number of sub problems
  - Order sub problems to avoid recomputation