CSE 421
Algorithms

Autumn 2019

Lecture 10

Minimum Spanning Trees

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Edge costs are assumed to be non-negative

Dijkstra's Algorithm Implementation and Runtime

S = { }; d[s] = 0; d[v] = infinity for v != s

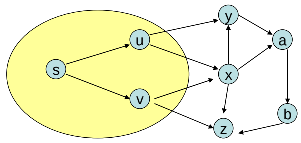
While S != V

Choose v in V-S with minimum d[v]

Add v to S

For each w in the neighborhood of v

d[w] = min(d[w], d[v] + c(v, w))



HEAP OPERATIONS

n Extract Mins

m Heap Updates

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Run Time

- Basic Heap Implementation
  - O(log n) extract min and update key
  - O((m + n) log n) run time
- Fancy data structures: Fibonacci Heaps
  - O(m + n log n)
- Dense graphs
  - $O(n^2)$

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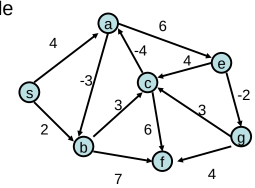
Shortest Paths

- Negative Cost Edges
  - Dijkstra's algorithm assumes positive cost edges
  - For some applications, negative cost edges make sense
  - Shortest path not well defined if a graph has a negative cost cycle



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Negative Cost Edge Preview

- Topological Sort can be used for solving the shortest path problem in directed acyclic graphs
- Bellman-Ford algorithm finds shortest paths in a graph with negative cost edges (or reports the existence of a negative cost cycle).

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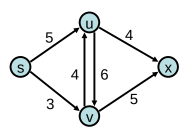
Bottleneck Shortest Path

- Define the bottleneck distance for a path to be the maximum cost edge along the path



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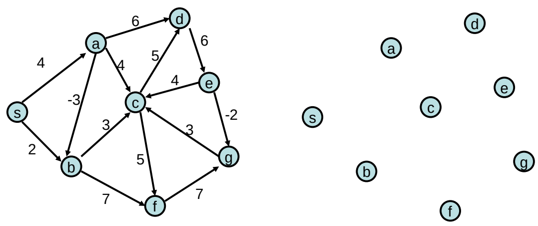
Compute the bottleneck shortest paths



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Dijkstra's Algorithm for Bottleneck Shortest Paths

S = { }; d[s] = negative infinity; d[v] = infinity for v != s

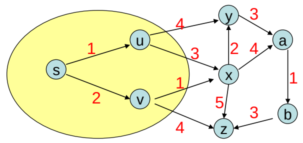
While S != V

Choose v in V-S with minimum d[v]

Add v to S

For each w in the neighborhood of v

d[w] = min(d[w], max(d[v], c(v, w)))



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Minimum Spanning Tree

- Introduce Problem
- Demonstrate three different greedy algorithms
- Provide proofs that the algorithms work

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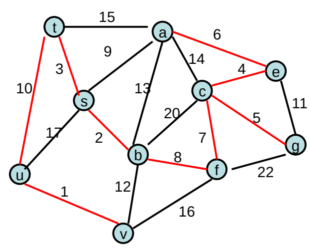
Minimum Spanning Tree



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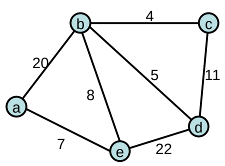
Greedy Algorithms for Minimum Spanning Tree

- Extend a tree by including the cheapest out going edge
- Add the cheapest edge that joins disjoint components
- Delete the most expensive edge that does not disconnect the graph



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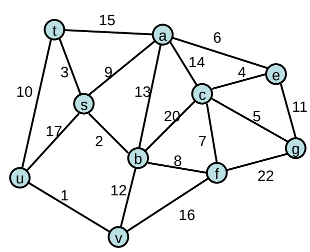
Greedy Algorithm 1 Prim's Algorithm

- Extend a tree by including the cheapest out going edge

Construct the MST with Prim's algorithm starting from vertex a

Label the edges in order of insertion



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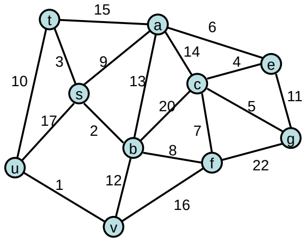
Greedy Algorithm 2
Kruskal's Algorithm

- Add the cheapest edge that joins disjoint components

Construct the MST with Kruskal's algorithm

Label the edges in order of insertion



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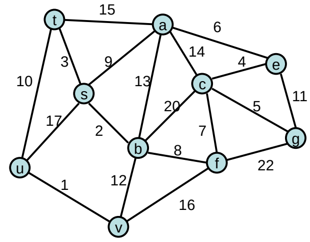
Greedy Algorithm 3
Reverse-Delete Algorithm

- Delete the most expensive edge that does not disconnect the graph

Construct the MST with the reverse-delete algorithm

Label the edges in order of removal



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Dijkstra's Algorithm for Minimum Spanning Trees

S = { }; d[s] = 0; d[v] = infinity for v != s

While S != V

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Add v to S

For each w in the neighborhood of v

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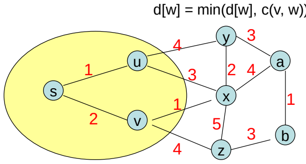

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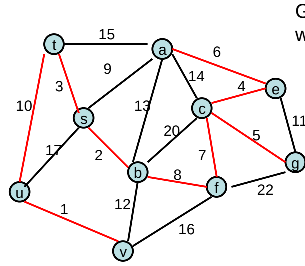
Minimum Spanning Tree

Undirected Graph G=(V,E) with edge weights



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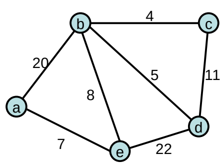
Greedy Algorithms for Minimum Spanning Tree

- [Prim] Extend a tree by including the cheapest out going edge
- [Kruskal] Add the cheapest edge that joins disjoint components
- [ReverseDelete] Delete the most expensive edge that does not disconnect the graph



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Why do the greedy algorithms work?

- For simplicity, assume all edge costs are distinct

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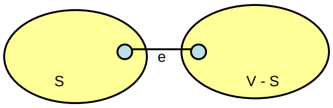
Edge inclusion lemma

- Let S be a subset of V, and suppose e = (u, v) is the minimum cost edge of E, with u in S and v in V-S
- e is in every minimum spanning tree of G
  - Or equivalently, if e is not in T, then T is not a minimum spanning tree



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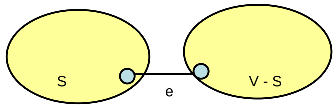
e is the minimum cost edge between S and V-S

Proof

- Suppose T is a spanning tree that does not contain e
- Add e to T, this creates a cycle
- The cycle must have some edge $e_1 = (u_1, v_1)$ with u_1 in S and v_1 in V-S



- $T_1 = T - \{e_1\} + \{e\}$ is a spanning tree with lower cost
- Hence, T is not a minimum spanning tree

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