Goals

Graphs: defns, examples, utility, terminology

Representation: input, internal

Traversal: Breadth- & Depth-first search

Five Graph Algorithms:
- Connected components
- Shortest Paths
- Topological sort
- Bipartiteness
- Articulation points
Objects & Relationships

The Kevin Bacon Game:
- Obj: Actors
- Rel: Two are related if they've been in a movie together

Exam Scheduling:
- Obj: Classes
- Rel: Two are related if they have students in common

Traveling Salesperson Problem:
- Obj: Cities
- Rel: Two are related if can travel directly between them
Graphs

An extremely important formalism for representing (binary) relationships
Objects: "vertices," aka "nodes"
Relationships between pairs:
  "edges," aka "arcs"
Formally, a graph $G = (V, E)$ is a pair of sets, $V$ the vertices and $E$ the edges
Undirected Graph $G = (V,E)$
Undirected Graph $G = (V,E)$
Undirected Graph $G = (V, E)$
Undirected Graph $G = (V, E)$
Undirected Graph $G = (V, E)$
Graphs don't live in Flatland

Geometrical drawing is mentally convenient, but mathematically irrelevant: 4 drawings, 1 graph.
Directed Graph $G = (V,E)$
Directed Graph $G = (V,E)$
Directed Graph $G = (V,E)$
Directed Graph $G = (V, E)$
Directed Graph $G = (V,E)$
Specifying undirected graphs as input

What are the vertices?
Explicitly list them:
{"A", "7", "3", "4"}

What are the edges?
Either, set of edges
{{A,3}, {7,4}, {4,3}, {4,A}}
Or, (symmetric) adjacency matrix:

\[
\begin{array}{cccc}
A & 7 & 3 & 4 \\
A & 0 & 1 & 1 \\
7 & 0 & 0 & 1 \\
3 & 1 & 0 & 1 \\
4 & 1 & 1 & 0 \\
\end{array}
\]
Specifying directed graphs as input

What are the vertices?
Explicitly list them:
{"A", "7", "3", "4"}

What are the edges?
Either, set of directed edges:
{(A,4), (4,7), (4,3), (4,A), (A,3)}
Or, (nonsymmetric) adjacency matrix:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>7</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>
# Vertices vs # Edges

Let $G$ be an undirected graph with $n$ vertices and $m$ edges. How are $n$ and $m$ related?

Since every edge connects two different vertices (no loops), and no two edges connect the same two vertices (no multi-edges), it must be true that:

$$0 \leq m \leq n(n-1)/2 = O(n^2)$$
More Cool Graph Lingo

A graph is called *sparse* if \( m \ll n^2 \), otherwise it is *dense*.

Boundary is somewhat fuzzy; \( O(n) \) edges is certainly sparse, \( \Omega(n^2) \) edges is dense.

Sparse graphs are common in practice.

E.g., all planar graphs are sparse \((m \leq 3n-6, \text{ for } n \geq 3)\).

Q: which is a better run time, \( O(n+m) \) or \( O(n^2) \)?

A: \( O(n+m) = O(n^2) \), but \( n+m \) usually way better!
Representing Graph $G = (V, E)$

Vertex set $V = \{v_1, \ldots, v_n\}$

Adjacency Matrix $A$

$A[i,j] = 1$ iff $(v_i, v_j) \in E$

Space is $n^2$ bits

Advantages:

$O(1)$ test for presence or absence of edges.

Disadvantages: inefficient for sparse graphs, both in storage and access

$$m << n^2$$
Representing Graph \( G=(V,E) \)
\( n \) vertices, \( m \) edges

Adjacency List:
\( O(n+m) \) words

Advantages:
- Compact for sparse graphs
- Easily see all edges

Disadvantages
- More complex data structure
- no \( O(1) \) edge test
Representing Graph $G=(V,E)$

$n$ vertices, $m$ edges

Adjacency List:
$O(n+m)$ words

Back- and cross pointers allow easier traversal and deletion of edges, if needed, but don't bother if not:
- more work to build,
- more storage overhead ($\sim 3m$ pointers)
Graph Traversal

Learn the basic structure of a graph
"Walk," via edges, from a fixed starting vertex s to all vertices reachable from s

Being orderly helps. Two common ways:
  Breadth-First Search
  Depth-First Search
Breadth-First Search

Completely explore the vertices in order of their distance from $s$

Naturally implemented using a queue
Breadth-First Search

Idea: Explore from s in all possible directions, layer by layer.

BFS algorithm.
\[ L_0 = \{ s \} \]
\[ L_1 = \text{all neighbors of } L_0 \]
\[ L_2 = \text{all nodes not in } L_0 \text{ or } L_1, \text{ and having an edge to a node in } L_1 \]
\[ L_{i+1} = \text{all nodes not in earlier layers, and having an edge to a node in } L_i \]

Theorem. For each \( i \), \( L_i \) consists of all nodes at distance (i.e., min path length) exactly \( i \) from \( s \).

Cor: There is a path from \( s \) to \( t \) iff \( t \) appears in some layer.
Graph Traversal: Implementation

Learn the basic structure of a graph
"Walk," via edges, from a fixed starting vertex $s$ to all vertices reachable from $s$

Three states of vertices
- undiscovered
- discovered
- fully-explored
BFS(s) Implementation

Global initialization: mark all vertices "undiscovered"

BFS(s)

mark s "discovered"
queue = { s }
while queue not empty
    u = remove_first(queue)
    for each edge {u,x}
        if (x is undiscovered)
            mark x discovered
            append x on queue
    mark u fully explored

Exercise: modify code to number vertices & compute level numbers
BFS(v)

Queue: 1
BFS\( (v) \)

Queue: 2 3
BFS(v)

Queue: 3 4
BFS($v$)

Queue: 4 5 6 7
BFS(v)

Queue: 5 6 7 8 9
BFS(v)

Queue: 10 11 12 13
BFS: Analysis, I

Global initialization: mark all vertices "undiscovered"

\[ O(n) \]

BFS(s)

\[ O(1) \]

mark s "discovered"

\[ O(1) \]

queue = \{ s \}

\[ O(n) \]

while queue not empty

\[ O(n) \]

\[ O(n^2) \]

= mark u fully explored

Simple analysis: 2 nested loops. Get worst-case number of iterations of each; multiply.
Above analysis correct, but pessimistic, assuming $G$ is sparse, edge list representation: can't have $\Omega(n)$ edges incident to each of $\Omega(n)$ distinct "u" vertices.

Alt, more global analysis:

Each edge is explored once from each end-point, so total runtime of inner loop is $O(m)$.

Total $O(n+m)$, $n = \# \text{ nodes}$, $m = \# \text{ edges}$

Exercise: extend algorithm and analysis to non-connected graphs
Properties of (Undirected) BFS($v$)

BFS($v$) visits $x$ if and only if there is a path in $G$ from $v$ to $x$.

Edges into then-undiscovered vertices define a tree – the "breadth first spanning tree" of $G$

Level $i$ in this tree are exactly those vertices $u$ such that the shortest path (in $G$, not just the tree) from the root $v$ is of length $i$.

All non-tree edges join vertices on the same or adjacent levels

not true of every spanning tree!
BFS Application: Shortest Paths

*Tree* (solid edges) gives shortest paths from start vertex.

Can label by distances from start. All edges connect same/adjacent levels.
**BFS Application: Shortest Paths**

*Tree* (solid edges) gives shortest paths from start vertex.

*Tree* (solid edges) gives shortest paths from start vertex.

- Can label by distances from start.
- All edges connect same/adjacent levels.
BFS Application: Shortest Paths

*Tree* (solid edges) gives shortest paths from start vertex.

Can label by distances from start, all edges connect same/adjacent levels.
BFS Application: Shortest Paths

Tree (solid edges) gives shortest paths from start vertex.

Can label by distances from start: all edges connect same/adjacent levels.
Why fuss about trees?

Trees are simpler than graphs
Ditto for algorithms on trees vs algs on graphs
So, this is often a good way to approach a graph problem: find a "nice" tree in the graph, i.e., one such that non-tree edges have some simplifying structure
E.g., BFS finds a tree s.t. level-jumps are minimized
DFS (below) finds a different tree, but it also has interesting structure…
Graph Search Application: Connected Components

Want to answer questions of the form:
given vertices $u$ and $v$, is there a path from $u$ to $v$?

Idea: create array $A$ such that

Q: Why not create 2-d array $Path[u,v]$?
Graph Search Application: Connected Components

initial state: all \( v \) undiscovered
for \( v = 1 \) to \( n \) do
  if state\((v)\) != \textit{fully-explored} then
    BFS\((v)\): setting \( A[u] \leftarrow v \) for each \( u \) found
    (and marking \( u \) discovered/fully-explored)
  endif
endif
endfor

Total cost: \( O(n+m) \)
  each edge is touched a constant number of times (twice)
  works also with DFS
3.4 Testing Bipartiteness
Bipartite Graphs

Def. An undirected graph $G = (V, E)$ is *bipartite (2-colorable)* if the nodes can be colored red or blue such that no edge has both ends the same color.

Applications.
- Stable marriage: men = red, women = blue
- Scheduling: machines = red, jobs = blue

"bi-partite" means "two parts." An equivalent definition: $G$ is bipartite if you can partition the node set into 2 parts (say, blue/red or left/right) so that all edges join nodes in different parts/no edge has both ends in the same part.

a bipartite graph
Testing Bipartiteness

Testing bipartiteness. Given a graph $G$, is it bipartite? Many graph problems become:
- easier if the underlying graph is bipartite (matching)
- tractable if the underlying graph is bipartite (independent set)
Before attempting to design an algorithm, we need to understand structure of bipartite graphs.

*a bipartite graph $G$*
An Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

Pf. Impossible to 2-color the odd cycle, let alone G.
Bipartite Graphs

Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).
Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

Pf. (i)
Suppose no edge joins two nodes in the same layer. By previous lemma, all edges join nodes on adjacent levels.

Bipartition:
red = nodes on odd levels, blue = nodes on even levels.
Bipartite Graphs

Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.
(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

Suppose $(x, y)$ is an edge & $x, y$ in same level $L_j$. Let $z = \text{lca}(x, y)$ be their lowest common ancestor in BFS tree. Let $L_i$ be level containing $z$. Consider cycle that takes edge from $x$ to $y$, then tree from $y$ to $z$, then tree from $z$ to $x$. Its length is $1 + (j-i) + (j-i)$, which is odd.
Obstruction to Bipartiteness

Cor: A graph $G$ is bipartite iff it contains no odd length cycle.

NB: the proof is algorithmic—it finds a coloring or odd cycle.

\begin{itemize}
\item[bipartite] (2-colorable)
\item[not bipartite] (not 2-colorable)
\end{itemize}
3.6 DAGs and Topological Ordering
Precedence Constraints

Precedence constraints. Edge \((v_i, v_j)\) means task \(v_i\) must occur before \(v_j\).

Many Applications

Course prerequisites: course \(v_i\) must be taken before \(v_j\)

Compilation: must compile module \(v_i\) before \(v_j\)

Computing workflow: output of job \(v_i\) is input to job \(v_j\)

Manufacturing or assembly: sand it before you paint it…

Spreadsheet evaluation order: if A7 is "=A6+A5+A4", evaluate them first
Directed Acyclic Graphs

Def. A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

Ex. Precedence constraints: edge \((v_i, v_j)\) means \(v_i\) must precede \(v_j\).

Def. A **topological order** of a directed graph \(G = (V, E)\) is an ordering of its nodes as \(v_1, v_2, \ldots, v_n\) so that for every edge \((v_i, v_j)\) we have \(i < j\).

E.g., \(\forall\) edge \((v_i, v_j)\), finish \(v_i\) before starting \(v_j\)

A DAG

A topological ordering of that DAG—
all edges oriented left-to-right
Lemma. If $G$ has a topological order, then $G$ is a DAG.

**Pf.** (by contradiction)

Suppose that $G$ has a topological order $v_1, \ldots, v_n$ and that $G$ also has a directed cycle $C$. Let $v_i$ be the lowest-indexed node in $C$, and let $v_j$ be the node just before $v_i$; thus $(v_j, v_i)$ is an edge. By our choice of $i$, we have $i < j$. On the other hand, since $(v_j, v_i)$ is an edge and $v_1, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction.

![Diagram of the topological order and directed cycle](image)
Directed Acyclic Graphs

Lemma (above).
If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?
Q. If so, how do we compute one?
Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)
Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u. Then, since u has at least one incoming edge (x, u), we can walk backward to x.
Repeat until we visit a node, say w, twice. Let C be the sequence of nodes encountered between successive visits to w. C is a cycle, contradicting acyclicity.
Directed Acyclic Graphs

Lemma. If $G$ is a DAG, then $G$ has a topological ordering.

Pf. (by induction on $n$)

- Base case: true if $n = 1$.
- Given DAG on $n > 1$ nodes, find a node $v$ with no incoming edges.
  $G - \{v\}$ is a DAG, since deleting $v$ cannot create cycles.
  By inductive hypothesis, $G - \{v\}$ has a topological ordering.
  Place $v$ first in topological ordering; then append nodes of $G - \{v\}$
in topological order. This is valid since $v$ has no incoming edges.

---

To compute a topological ordering of $G$:
Find a node $v$ with no incoming edges and order it first
Delete $v$ from $G$
Recursively compute a topological ordering of $G-\{v\}$
    and append this order after $v$
Topological Ordering Algorithm: Example

Topological order:
Topological Ordering Algorithm: Example

Topological order: \( v_1 \)
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6$
Topological Ordering Algorithm: Example

Topological order: \( v_1, v_2, v_3, v_4, v_5, v_6, v_7 \).
Topological Sorting Algorithm

Maintain the following:
- count[w] = (remaining) number of incoming edges to node w
- S = set of (remaining) nodes with no incoming edges

Initialization:
- count[w] = 0 for all w
- count[w]++ for all edges (v,w)
- S = S ∪ {w} for all w with count[w]==0

Main loop:
- while S not empty
  - remove some v from S
  - make v next in topo order
  - for all edges from v to some w
    - count[w]--
    - if count[w] == 0 then add w to S

Correctness: clear, I hope

Time: O(m + n) (assuming edge-list representation of graph)
Depth-First Search
Depth-First Search

Follow the first path you find as far as you can go
When you reach a dead end, back up to last unexplored edge, then go as far you can. Etc.

Naturally implemented using recursive calls or a stack
DFS(v) – Recursive version

Global Initialization:
   for all nodes v, v.dfs# = -1  // mark v "undiscovered"
   dfscounter = 0

DFS(v)
   v.dfs# = dfscounter++  // v "discovered", number it
   for each edge (v,x)
      if (x.dfs# = -1)  // tree edge (x previously undiscovered)
         DFS(x)
   else …  // code for back-, fwd-, parent-
     // edges, if needed; mark v
   // "completed," if needed
Why fuss about trees (again)?

BFS tree $\neq$ DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – *only descendant/ancestor*

Proof below
Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack (Edge list):
A (B, J)
Suppose edge lists at each vertex are sorted alphabetically.

**DFS(A)**

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack:
- (Edge list)
  - A (B,J)
  - B (A,C,J)

Diagram:
- A,1
- B,2
- C
- D
- E
- F
- G
- H
- I
- J
- K
- L
- M
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack:
(Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
**DFS(A)**

Suppose edge lists at each vertex are sorted alphabetically.

**Color code:**
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**Call Stack:**
- (Edge list)
  - A (B,J)
  - B (A,C,J)
  - C (B,D,G,H)
  - D (C,E,F)
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically.

Color code:
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Call Stack:
- (Edge list)
- A (B, J)
- B (A, C, J)
- C (B, D, G, H)
- D (C, E, F)
- E (D, F)
Suppose edge lists at each vertex are sorted alphabetically.

**Color code:**
- **undiscovered**
- **discovered**
- **fully-explored**

**Call Stack:**
(Edge list)
- A (B, J)
- B (A, C, J)
- C (B, D, G, H)
- D (C, E, F)
- E (D, F)
- F (D, E, G)
Suppose edge lists at each vertex are sorted alphabetically.

DFS(A)

Call Stack:
- A (B,J)
- B (A,C,J)
- C (B,D,G,H)
- D (C,E,F)
- E (D,F)
- F (E,G)
- G (C,F)

Color code:
- undiscovered
- discovered
- fully-explored
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- undiscovered
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Call Stack:
- (Edge list)
  - A (B,J)
  - B (A,C,J)
  - C (B,D,G,H)
  - D (C,F,E)
  - E (D,F)
  - F (D,E,G)
  - G (C,F)

- A,1
- B,2
- C,3
- D,4
- E,5
- F,6
- G,7
- H
- I
- J
- K
- L
- M
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack:
(Edge list)
- A (B,J)
- B (A,C,J)
- C (B,D,G,H)
- D (C,F,F)
- E (D,F)
- F (E,G)

A,1
B,2
C,3
G,7
D,4
F,6
E,5
Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack:
- (Edge list)
  - A (B, J)
  - B (A, C, J)
  - C (B, D, G, H)
  - D (C, E, F)
  - E (D, F)

DFS(A)
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically.

Call Stack:
(Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
D (C,E,F)

Color code:
- undiscovered
- discovered
- fully-explored
Suppose edge lists at each vertex are sorted alphabetically.

Call Stack:
- (Edge list)
  - A (B,J)
  - B (A,C,J)
  - C (B,D,G,H)

Color code:
- undiscovered
- discovered
- fully-explored

DFS(A)
Suppose edge lists at each vertex are sorted alphabetically.

**DFS(A)**

Color code:
- **undiscovered**
- **discovered**
- **fully-explored**

Call Stack:
- (Edge list)
  - A (B, J)
  - B (A, C, J)
  - C (B, D, G, H)

```
A,1
B,2
C,3
D,4
E,5
F,6
G,7
H
I
J
K
L
M
```
Suppose edge lists at each vertex are sorted alphabetically.

The diagram illustrates a DFS traversal starting from vertex A. Each vertex is color-coded to indicate its status:
- **undiscovered**
- **discovered**
- **fully-explored**

The call stack shows the edge lists encountered during the traversal:
- A → (B, J)
- B → (A, C, J)
- C → (B, D, G, H)
- H → (C, I, J)

The edge lists are as follows:
- A: 1
- B: 2
- C: 3
- D: 4
- E: 5
- F: 6
- G: 7
- H: 8
- K
- L
- M

The diagram also shows the order of discovery:
- A (discovered)
- B (discovered)
- C (discovered)
- H (fully-explored)
- J (fully-explored)
- K
- L
- M
Suppose edge lists at each vertex are sorted alphabetically.

**DFS(A)**

Color code:
- **undiscovered**
- **discovered**
- **fully-explored**

Call Stack:
(Edge list)

- A (B, J)
- B (A, C, J)
- C (B, D, G, H)
- H (C, I, J)
- I (H)

- L
- K
- M
Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack:
- (Edge list)
  - A (B,J)
  - B (A,C,J)
  - C (B,D,G,H)
  - H (C,I,J)
  - I (H)
  - K
  - L
  - M
Suppose edge lists at each vertex are sorted alphabetically.

**DFS(A)**

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack: (Edge list)
- A (B,J)
- B (A,C,J)
- C (B,D,G,H)
- H (C,I,J)

Diagram:
- A (1)
- B (2)
- C (3)
- D (4)
- E (5)
- F (6)
- G (7)
- H (8)
- J
- K
- L
- M

Edge connections:
- A → B
- B → A, C, J
- C → B, D, G, H
- H → C, I, J
- J
- K
- L
- M

Disconnected vertices:
Suppose edge lists at each vertex are sorted alphabetically.

Call Stack:
(Edge list)
A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
Suppose edge lists at each vertex are sorted alphabetically.

**DFS(A)**

Color code:
- **undiscovered**
- **discovered**
- **fully-explored**

**Call Stack:**
- (Edge list)
  - A (B,J)
  - B (A,C,J)
  - C (B,D,G,H)
  - H (C,I,J)
  - J (A,B,H,K,L)
  - K (J,L)

**DFS Algorithm**

1. **Start** with vertex A (undiscovered).
3. Explore each vertex's edge list in alphabetical order.
4. Continue until all vertices are fully-explored.
Suppose edge lists at each vertex are sorted alphabetically.

**DFS(A)**

Call Stack:
(Edge list)

A (B,J)
B (A,C,J)
C (B,D,G,H)
H (C,I,J)
J (A,B,H,K,L)
K (J,L)
L (J,K,M)

Color code:
- undiscovered
- discovered
- fully-explored
Suppose edge lists at each vertex are sorted alphabetically.

DFS(A)

Call Stack:
(Edge list)

- A (B, J)
- B (A, C, J)
- C (B, D, G, H)
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- J (A, B, H, K, L)
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- L (J, K, M)
- M (L)
Suppose edge lists at each vertex are sorted alphabetically.

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- J (A,B,H,K,L)
- K (J,L)
- L (K)
- M (K)
- E (M,5)
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DFS(A)

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack:
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- M (L)

\[ \text{DFS(A)} \]
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**DFS(A)**

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- fully-explored

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**Suppose edge lists at each vertex are sorted alphabetically.**
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Call Stack:
(Edge list)
A (B, J)
B (A, C, J)
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically.

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack:
(Edge list)
A (B, J)
Suppose edge lists at each vertex are sorted alphabetically.

**DFS(A)**

Color code:
- undiscovered
- discovered
- fully-explored

Call Stack:
- (Edge list)
- A (B, J)

Diagram:
- A,1
  - B,2
    - C,3
    - D,4
      - E,5
    - G,7
  - H,8
  - J,10
    - K,11
    - L,12
    - M,13
DFS(A)

Suppose edge lists at each vertex are sorted alphabetically.

Call Stack: (Edge list)

TA-DA!!
DFS(A)

Edge code:

Tree edge

Back edge

A,1

B,2

C,3

D,4

E,5

F,6

G,7

H,8

I,9

J,10

K,11

L,12

M,13
DFS(A)

Edge code:
- Tree edge
- Back edge

Diagram showing a tree with nodes labeled from A to M and numbers 1 to 13. The edge types are indicated by solid and dashed lines.
DFS(A)

Edge code:
- Tree edge
- Back edge
DFS(A)

Edge code:
- Tree edge
- Back edge
DFS(A)
DFS(A)

Edge code:
Tree edge
Back edge
No Cross Edges!
Properties of (Undirected) DFS(v)

Like BFS(v):

- DFS(v) visits x if and only if there is a path in G from v to x (through previously unvisited vertices)
- Edges into then-undiscovered vertices define a tree – the "depth first spanning tree" of G

Unlike the BFS tree:

- the DF spanning tree isn't minimum depth
- its levels don't reflect min distance from the root
- non-tree edges never join vertices on the same or adjacent levels

BUT…
Non-tree edges

All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

No cross edges!
Why fuss about trees (again)?

As with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – only descendant/ancestor
A simple problem on trees

**Given:** tree T, a value $L(v)$ defined for every vertex $v$ in T

**Goal:** find $M(v)$, the min value of $L(v)$ anywhere in the subtree rooted at $v$ (including $v$ itself).

**How?** Depth first search, using:

$$M(v) = \begin{cases} L(v) & \text{if } v \text{ is a leaf} \\ \min(L(v), \min_{\text{a child of } v} M(w)) & \text{otherwise} \end{cases}$$
Application: Articulation Points

A node in an undirected graph is an *articulation point* if it disconnects the graph (or, more generally, increases the number of connected components).

Articulation points represent, e.g.:
- vulnerabilities in a network – single points whose failure would split the network into 2 or more disconnected components
- bottlenecks to information flow in a network
Identifying key proteins on the anthrax predicted network
Articulation Points

articulation point
iff its removal disconnects the graph
Articulation Points
Simple Case: Artic. Pts in a tree

Leaves – never articulation points
Internal nodes – always articulation points
Root – articulation point if and only if two or more children

Non-tree: extra edges remove some articulation points (which ones?)
Articulation Points from DFS

Root node is an articulation point iff it has more than one child.

Leaf is never an articulation point.

Non-leaf, non-root node $u$ is an articulation point iff

$\exists$ some child $y$ of $u$ s.t. no non-tree edge goes above $u$ from $y$ or below.

If $u$'s removal does NOT separate $x$, there must be an exit from $x$'s subtree. How? Via back edge.
Articulation Points: the "LOW" function

Definition: LOW(v) is the lowest dfs# of any vertex that is either in the dfs subtree rooted at v (including v itself) or directly connected to a vertex in that subtree by one back edge.

Key idea 1: if some child x of v has LOW(x) ≥ dfs#(v) then v is an articulation point (excl. root)

Key idea 2: LOW(v) =
\[
\min ( \{ \text{dfs#}(v) \} \cup \{ \text{LOW}(w) \mid w \text{ a child of } v \} \cup \{ \text{dfs#}(x) \mid \{v,x\} \text{ is a back edge from } v \} )
\]
DFS(v) for Finding Articulation Points

Global initialization: dfscounter = 0; v.dfs# = -1 for all v.

DFS(v)

v.dfs# = dfscounter++

v.low = v.dfs# // initialization

for each edge {v,x}

if (x.dfs# == -1) // x is undiscovered

DFS(x)

v.low = min(v.low, x.low)

if (x.low >= v.dfs#)

print "v is art. pt., separating x"

else if (x is not v's parent)

v.low = min(v.low, x.dfs#)

Equiv: "if( {v,x} is a back edge)"

Why?
Articulation Points

A
B
C
D
E
F
G
H
I
J
K
L
M

Vertex | DFS # | Low
--- | --- | ---
A  |   |   
B  |   |   
C  |   |   
D  |   |   
E  |   |   
F  |   |   
G  |   |   
H  |   |   
I  |   |   
J  |   |   
K  |   |   
L  |   |   
M  |   |   

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Articulation Points

<table>
<thead>
<tr>
<th>Vertex</th>
<th>DFS #</th>
<th>Low</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>E</td>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>F</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>G</td>
<td>9</td>
<td>9</td>
</tr>
<tr>
<td>H</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>I</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>J</td>
<td>11</td>
<td>10</td>
</tr>
<tr>
<td>K</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>L</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>M</td>
<td>13</td>
<td>13</td>
</tr>
</tbody>
</table>
Summary

Graphs – abstract relationships among pairs of objects
Terminology – node/vertex/vertices, edges, paths, multi-edges, self-loops, connected
Representation – edge list, adjacency matrix
Nodes vs Edges – $m = O(n^2)$, often less
BFS – Layers, queue, shortest paths, all edges go to same or adjacent layer
DFS – recursion/stack; all edges ancestor/descendant
Algorithms – connected components, shortest path, bipartiteness, topological sort, articulation points