CSE 417: Algorithms and Computational Complexity **Winter 2012**

Writing Induction Proofs

Many of the proofs presented in class and asked for in the homework require induction. Here is a short guide to writing such proofs.

First, we outline in abstract terms the form that induction proofs should take. Unless you are very experienced writing inductive proofs, you should follow this outline explicitly to make sure that you don't miss a critical step. The proofs in the book and the homework solutions don't always follow this form explicitly, but they do at least touch on each part in some way.

Second, we provide some examples of inductive proofs that follow the structure outlined in the first part.

Finally, we provide some example problems for practice. We don't have solutions, but you can feel free to bring your solutions into office hours to talk through them with any of the TAs or instructors.

General Form of an Inductive Proof

Assume we wish to prove a statement $P(n)$ where $n \geq c$, where c is a natural number, often (but not always) 0 or 1. For example, our statement might be "A full binary trees of depth $n \geq 0$ has exactly $2^{n+1} - 1$ nodes" or "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, for all $n \geq 1$". The basic skeleton of an inductive proof is the following:

1. **State what we want to prove**: $P(n)$ for all $n \geq c$, $c \geq 0$ by induction on n. The actual words that are used here will depend on the form of the claim. (See the examples below.)

2. **Base case:** Prove $P(c)$. This is usually easy to prove. It can be done by considering cases or explicitly computing values.

3. **Inductive hypothesis:** Let $k \geq c$ be an arbitrary integer. Assume $P(k)$ is true. (Some induction proofs require that we assume $P(n)$ is true for all $c \leq n \leq k$. That proof technique is called *Strong Induction*.)

4. **Inductive step** Prove $P(k+1)$, assuming that $P(k)$ is true. This is often the most involved part of the proof. Apart from proving the base case, it is usually the only part that is not boilerplate.

5. Apply the **Induction rule**: If have shown that $P(c)$ holds and that for all integers $k \geq c$, assuming $P(k)$ implies $P(k+1)$, then $P(n)$ holds for all $n \geq c$. In symbols, this can be written as $(P(c) \& P(k) \to P(k+1) \forall k \geq c) \to P(n) \forall n \geq c$.

Examples

Example 1

Prove $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, for all $n \geq 1$.

Base case: $\sum_{i=1}^{1} i = 1$ and $\frac{1\cdot(1+1)}{2} = 1$. $P(1)$ is true.

Inductive hypothesis: Let $k \geq c$ be an arbitrary integer. Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$.

Inductive step: We want to prove $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

$$
\begin{aligned}
\sum_{i=1}^{k+1} i &= 1 + 2 + \ldots k + (k+1) \\
&= (1 + 2 + \ldots k) + (k+1) \\
&= \frac{k(k+1)}{2} + k + 1 \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

By the **Induction rule**, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, for all $n \geq 1$.

Example 2

Prove that a full binary trees of depth $n \geq 0$ has exactly $2^{n+1} - 1$ nodes.

Base case: Let T be a full binary tree of depth 0. Then T has exactly one node. Then $P(0)$ is true.

Inductive hypothesis: Let T be a full binary tree of depth k. Then T has exactly $2^{k+1} - 1$ nodes.

Inductive step: Let T' be a full binary tree of depth $k+1$. Consider the subtrees rooted at the children of the root T_L and T_R. Both of these trees are full binary trees of depth k. Then the numbers of nodes in T_L is $2^{k+1} - 1$. Same for T_R. Then the total number of nodes of T is $2(2^{k+1} - 1) + 1 = 2^{k+2} - 1$.

By the **Induction rule**, a full binary trees of depth $n \geq 0$ has exactly $2^{n+1} - 1$ nodes.

1 Exercises

Exercise 1

Prove that for all $n \geq 0$,

$$\sum_{i=0}^{n} i2^i = 2 + (n-1)2^{n+1} \quad .$$

Exercise 2

Let $P(n)$ be that any n lines, where no two are parallel and no three pass through the same point, divide the plane into $n^2 + 1$ regions. What is wrong with the following inductive proof? It is not sufficient to give a counterexample to the given theorem. Rather, you must find the flaw in the proof.

Base case: One line divides the plane into 2 regions and $1 = 1^2 + 1$. Hence $P(1)$ is true.

Inductive hypothesis: Assume that k lines divide the plane into $k^2 + 1$ regions.

Inductive step: Note that $(k+1)^2+1 = k^2+1+2k+1$. So adding the $(k+1)$st line creates $2k+1$ new regions. Hence the number of regions with $k+1$ lines is $k^2+1+2k+1 = (k+1)^2+1$.

By the **Induction rule**, since $P(1)$ holds we have shown that $P(k) \rightarrow P(k+1)$, we conclude that any n lines divide the plane into $n^2 + 1$ regions.

(Exercise written by Sally Goldman.)

Exercise 3

The following claim is similar to Claim 4.9.a on page 129 of the book regarding inversions in a schedule, which we covered in our second lecture on greedy algorithms. This exercise may help you to understand that proof better.

Claim: Suppose there are n men and women standing in a line. If the person at the head of the line is a man and the person at the end of the line is a woman, then somewhere in the line, there must be a man standing next to a woman.

Prove this claim by induction on n.