Goals

Graphs: defns, examples, utility, terminology
Representation: input, internal
Traversal: Breadth- & Depth-first search
Three Algorithms:
  Connected components
  Bipartiteness
  Topological sort
Meg Ryan was in "French Kiss" with Kevin Kline

Meg Ryan was in "Sleepless in Seattle" with Tom Hanks

Kevin Bacon was in "Apollo 13" with Tom Hanks
Objects & Relationships

The Kevin Bacon Game:
  Actors
  Two are related if they’ve been in a movie together

Exam Scheduling:
  Classes
  Two are related if they have students in common

Traveling Salesperson Problem:
  Cities
  Two are related if can travel \textbf{directly} between them
Graphs

An extremely important formalism for representing (binary) relationships

Objects: “vertices”, aka “nodes”

Relationships between pairs: “edges”, aka “arcs”

Formally, a graph $G = (V, E)$ is a pair of sets, $V$ the vertices and $E$ the edges
Undirected Graph \( G = (V,E) \)
Undirected Graph $G = (V, E)$
Undirected Graph \( G = (V,E) \)
Undirected Graph $G = (V, E)$
Undirected Graph \( G = (V,E) \)
Graphs don’t live in Flatland

Geometrical drawing is mentally convenient, but mathematically irrelevant: 4 drawings, 1 graph.
Directed Graph $G = (V,E)$
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Directed Graph $G = (V, E)$
Specifying undirected graphs as input

What are the vertices?
Explicitly list them:
{“A”, “7”, “3”, “4”}

What are the edges?
Either, set of edges
{{A,3}, {7,4}, {4,3}, {4,A}}
Or, (symmetric) adjacency matrix:
Specifying directed graphs as input

What are the vertices?
Explicitly list them:
{“A”, “7”, “3”, “4”}

What are the edges?
Either, set of directed edges:
{(A,4), (4,7), (4,3), (4,A), (A,3)}
Or, (nonsymmetric) adjacency matrix:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>7</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
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<td>3</td>
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<td>4</td>
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</tbody>
</table>
Let G be an undirected graph with n vertices and m edges. How are n and m related?

Since every edge connects two different vertices (no loops), and no two edges connect the same two vertices (no multi-edges), it must be true that:

\[ 0 \leq m \leq \frac{n(n-1)}{2} = O(n^2) \]
More Cool Graph Lingo

A graph is called \textit{sparse} if \( m \ll n^2 \), otherwise it is \textit{dense).

Boundary is somewhat fuzzy; \( O(n) \) edges is certainly sparse, \( \Omega(n^2) \) edges is dense.

Sparse graphs are common in practice

E.g., all planar graphs are sparse (\( m \leq 3n-6 \), for \( n \geq 3 \))

Q: which is a better run time, \( O(n+m) \) or \( O(n^2) \)?

A: \( O(n+m) = O(n^2) \), but \( n+m \) usually way better!
Representing Graph \( G = (V,E) \)

Vertex set \( V = \{v_1, \ldots, v_n\} \)

Adjacency Matrix \( A \)

\[ A[i,j] = 1 \text{ iff } (v_i,v_j) \in E \]

Space is \( n^2 \) bits

Advantages:

\( O(1) \) test for presence or absence of edges.

Disadvantages: inefficient for sparse graphs, both in storage and access

\[ \text{m} \ll n^2 \]
Representing Graph $G = (V, E)$
$n$ vertices, $m$ edges

Adjacency List:
$O(n+m)$ words

Advantages:
- Compact for sparse graphs
- Easily see all edges

Disadvantages
- More complex data structure
- no $O(1)$ edge test
Representing Graph $G=(V,E)$

$n$ vertices, $m$ edges

Adjacency List:
$O(n+m)$ words

Back- and cross pointers more work to build, but allow easier traversal and deletion of edges, if needed, (don’t bother if not)
Graph Traversal

Learn the basic structure of a graph
“Walk,” via edges, from a fixed starting vertex $s$ to all vertices reachable from $s$

Being orderly helps. Two common ways:

Breadth-First Search
Depth-First Search
Breadth-First Search

Completely explore the vertices in order of their distance from $s$

Naturally implemented using a queue
Breadth-First Search

Idea: Explore from s in all possible directions, layer by layer.

BFS algorithm.
- $L_0 = \{ s \}.$
- $L_1 =$ all neighbors of $L_0.$
- $L_2 =$ all nodes not in $L_0$ or $L_1$, and having an edge to a node in $L_1$.
- $L_{i+1} =$ all nodes not in earlier layers, and having an edge to a node in $L_i$.

Theorem. For each $i$, $L_i$ consists of all nodes at distance (i.e., min path length) exactly $i$ from $s$.

Cor: There is a path from $s$ to $t$ iff $t$ appears in some layer.
Graph Traversal: Implementation

Learn the basic structure of a graph
“Walk,” via edges, from a fixed starting vertex $s$ to all vertices reachable from $s$

Three states of vertices
  undiscovered
  discovered
  fully-explored
BFS(s) Implementation

Global initialization: mark all vertices "undiscovered"

BFS(s)

mark s "discovered"
queue = { s }
while queue not empty
    u = remove_first(queue)
    for each edge {u,x}
        if (x is undiscovered)
            mark x discovered
            append x on queue
    mark u fully explored

Exercise: modify code to number vertices & compute level numbers
BFS(v)
BFS(ν)

Queue: 3 4
BFS(v)

Queue: 5 6 7 8 9
BFS(v)

Queue: 8 9 10 11
BFS(v)

Queue: 10 11 12 13
BFS(v)
**BFS(s) Implementation**

Global initialization: mark all vertices "undiscovered"

BFS(s)
  mark s "discovered"
  queue = { s }
  while queue not empty
    u = remove_first(queue)
    for each edge {u,x}
      if (x is undiscovered)
        mark x discovered
        append x on queue
    mark u fully explored

Exercise: modify code to number vertices & compute level numbers
BFS analysis

Each vertex is added to/removed from queue at most once

Each edge is explored once from each end-point

Total cost $O(m)$, $m =$ # of edges

Exercise: extend algorithm and analysis to non-connected graphs
Properties of (Undirected) BFS(v)

BFS(v) visits x if and only if there is a path in G from v to x.

Edges into then-undiscovered vertices define a tree – the "breadth first spanning tree" of G.

Level i in this tree are exactly those vertices u such that the shortest path (in G, not just the tree) from the root v is of length i.

All non-tree edges join vertices on the same or adjacent levels.

not true of every spanning tree!
BFS Application: Shortest Paths

*Tree* (solid edges) gives shortest paths from start vertex.

Can label by distances from start, all edges connect same/adjacent levels.
**BFS Application: Shortest Paths**

*Tree* (solid edges) gives shortest paths from start vertex.

- Can label by distances from start.
- All edges connect same/adjacent levels.
BFS Application: Shortest Paths

*Tree* (solid edges) gives shortest paths from start vertex

can label by distances from start

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BFS Application: Shortest Paths

Tree (solid edges) gives shortest paths from start vertex

can label by distances from start
all edges connect same/adjacent levels
Why fuss about trees?

Trees are simpler than graphs
Ditto for algorithms on trees vs algs on graphs
So, this is often a good way to approach a graph problem: find a “nice” tree in the graph, i.e., one such that non-tree edges have some simplifying structure
E.g., BFS finds a tree s.t. level-jumps are minimized
DFS (next) finds a different tree, but it also has interesting structure…
Graph Search Application: Connected Components

Want to answer questions of the form:
given vertices u and v, is there a path from u to v?

Idea: create array A such that
A[u] = smallest numbered vertex that is connected to u. Question reduces to whether A[u]=A[v]?

Q: Why not create 2-d array Path[u,v]?
Graph Search Application: Connected Components

initial state: all $v$ undiscovered

for $v = 1$ to $n$ do
  if state($v$) != fully-explored then
    BFS($v$): setting $A[u] \leftarrow v$ for each $u$ found
    (and marking $u$ discovered/fully-explored)
  endif
endfor

Total cost: $O(n+m)$

each edge is touched a constant number of times (twice)
works also with DFS
3.4 Testing Bipartiteness
Def. An undirected graph $G = (V, E)$ is \textit{bipartite} if the nodes can be colored red or blue such that every edge has one red and one blue end.

**Applications.**
- Stable marriage: men = red, women = blue
- Scheduling: machines = red, jobs = blue

“bi-partite” means “two parts.” An equivalent definition: $G$ is bipartite if you can partition the node set into 2 parts (say, blue/red or left/right) so that all edges join nodes in different parts/no edge has both ends in the same part.
Testing bipartiteness. Given a graph $G$, is it bipartite?

Many graph problems become:
- easier if the underlying graph is bipartite (matching)
- tractable if the underlying graph is bipartite (independent set)

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.
Lemma. If a graph $G$ is bipartite, it cannot contain an odd length cycle.

Pf. Impossible to 2-color the odd cycle, let alone $G$. 
Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.
(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).
Lemma. Let $G$ be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.
(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

Pf. (i)
Suppose no edge joins two nodes in the same layer. By previous lemma, all edges join nodes on adjacent levels.

Bipartition:
red = nodes on odd levels,
blue = nodes on even levels.

Case (i)
Lemma. Let $G$ be a connected graph, and let $L_0$, ..., $L_k$ be the layers produced by BFS starting at node $s$. Exactly one of the following holds.

(i) No edge of $G$ joins two nodes of the same layer, and $G$ is bipartite.

(ii) An edge of $G$ joins two nodes of the same layer, and $G$ contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

Suppose $(x, y)$ is an edge & $x, y$ in same level $L_j$. Let $z =$ their lowest common ancestor in BFS tree. Let $L_i$ be level containing $z$.

Consider cycle that takes edge from $x$ to $y$, then tree from $y$ to $z$, then tree from $z$ to $x$. Its length is $1 + (j-i) + (j-i)$, which is odd.
Obstruction to Bipartiteness

Cor: A graph $G$ is bipartite iff it contains no odd length cycle.

NB: the proof is algorithmic-- in a non-bipartite graph, it finds an odd cycle.
3.6 DAGs and Topological Ordering
Precedence Constraints

Precedence constraints. Edge \((v_i, v_j)\) means task \(v_i\) must occur before \(v_j\).

Applications

Course prerequisites: course \(v_i\) must be taken before \(v_j\)

Compilation: must compile module \(v_i\) before \(v_j\)

Job Workflow: output of job \(v_i\) is part of input to job \(v_j\)

Manufacturing or assembly: sand it before you paint it…

Spreadsheet evaluation: cell \(v_j\) depends on \(v_i\)
Directed Acyclic Graphs

Def. A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

Ex. Precedence constraints: edge \((v_i, v_j)\) means \(v_i\) must precede \(v_j\).

Def. A topological order of a directed graph \(G = (V, E)\) is an ordering of its nodes as \(v_1, v_2, \ldots, v_n\) so that for every edge \((v_i, v_j)\) we have \(i < j\).
Directed Acyclic Graphs

Lemma. If G has a topological order, then G is a DAG.

Pf. (by contradiction)
Suppose that G has a topological order $v_1, \ldots, v_n$
and that G also has a directed cycle C.
Let $v_a$ be the lowest-indexed node in C, and let $v_b$ be the node just
before $v_a$ in the cycle; thus $(v_b, v_a)$ is an edge.
By our choice of $a$, we have $a < b$.
On the other hand, since $(v_b, v_a)$ is an edge and $v_1, \ldots, v_n$ is a
topological order, we must have $b < a$, a contradiction. □
Directed Acyclic Graphs

Lemma.
If $G$ has a topological order, then $G$ is a DAG.

Q. Does every DAG have a topological ordering?

Q. If so, how do we compute one?
Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)
Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u. Then, since u has at least one incoming edge (x, u), we can walk backward to x.
Repeat until we visit a node, say w, twice. Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.

Why must this happen?
Directed Acyclic Graphs

Lemma. If $G$ is a DAG, then $G$ has a topological ordering.

Pf. (by induction on $n$)

Base case: true if $n = 1$.

Given DAG on $n > 1$ nodes, find a node $v$ with no incoming edges. $G - \{v\}$ is a DAG, since deleting $v$ cannot create cycles.

By inductive hypothesis, $G - \{v\}$ has a topological ordering.

Place $v$ first in topological ordering; then append nodes of $G - \{v\}$ in topological order. This is valid since $v$ has no incoming edges. □

To compute a topological ordering of $G$:
Find a node $v$ with no incoming edges and order it first
Delete $v$ from $G$
Recursively compute a topological ordering of $G - \{v\}$
and append this order after $v$
Topological Ordering Algorithm: Example

Topological order:
Topological Ordering Algorithm: Example

Topological order: \( v_1 \)
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6$
Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$. 
Topological Sorting Algorithm

Maintain the following:
- \( \text{count}[w] \) = (remaining) number of incoming edges to node \( w \)
- \( S \) = set of (remaining) nodes with no incoming edges

Initialization:
- \( \text{count}[w] = 0 \) for all \( w \)
- \( \text{count}[w]++ \) for all edges \((v,w)\)
- \( S = S \cup \{w\} \) for all \( w \) with \( \text{count}[w] = 0 \)

Main loop:
- while \( S \) not empty
  - remove some \( v \) from \( S \)
  - make \( v \) next in topo order
  - for all edges from \( v \) to some \( w \)
    - decrement \( \text{count}[w] \)
    - add \( w \) to \( S \) if \( \text{count}[w] \) hits 0

Correctness: clear, I hope
Time: \( O(m + n) \) (assuming edge-list representation of graph)
Depth-First Search

Follow the first path you find as far as you can go
Back up to last unexplored edge when you reach a dead end, then go as far you can

Naturally implemented using recursive calls or a stack
DFS(v) – Recursive version

Global Initialization:
   for all nodes v, v.dfs# = -1 // mark v "undiscovered"
dfscounter = 0

DFS(v)
   v.dfs# = dfscounter++ // v “discovered”, number it
   for each edge (v,x)
      if (x.dfs# = -1)       // tree edge (x previously undiscovered)
         DFS(x)
      else …                 // code for back-, fwd-, parent,
                                 // edges, if needed
                                 // mark v “completed,” if needed
Non-tree edges

All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

No cross edges!
Why fuss about trees (again)?

BFS tree ≠ DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are “simple” – only descendant/ancestor
Summary

Graphs – abstract relationships among pairs of objects
Terminology – node/vertex/vertices, edges, paths, multi-edges, self-loops, connected
Representation – edge list, adjacency matrix
Nodes vs Edges – $m = O(n^2)$, often less
BFS – Layers, queue, shortest paths, all edges go to same or adjacent layer
DFS – recursion/stack; all edges ancestor/descendant
Algorithms – connected components, bipartiteness, topological sort