Chapter 6
Dynamic Programming

Algorithmic Paradigms

Greedy. Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer. Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

Dynamic Programming History

Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

Etymology.
- Dynamic programming = planning over time.
- Secretary of Defense was hostile to mathematical research.
- Bellman sought an impressive name to avoid confrontation.
- “It’s impossible to use dynamic in a pejorative sense”
- “Something not even a Congressman could object to”


Dynamic Programming Applications

Areas.
- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, systems, …

Some famous dynamic programming algorithms.
- Viterbi for hidden Markov models.
- Unix diff for comparing two files.
- Smith-Waterman for sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.
6.1 Weighted Interval Scheduling

Weighted interval scheduling problem.
- Job \( j \) starts at \( s_j \), finishes at \( f_j \), and has weight or value \( v_j \).
- Two jobs compatible if they don’t overlap.
- Goal: find maximum weight subset of mutually compatible jobs.

Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.
- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.

Weighted Interval Scheduling

Notation. Label jobs by finishing time: \( f_1 \leq f_2 \leq \ldots \leq f_n \).
Def. \( p(j) \) = largest index \( i < j \) such that job \( i \) is compatible with \( j \).

Ex: \( p(8) = 5, p(7) = 3, p(2) = 0 \).
Dynamic Programming: Binary Choice

Notation. \( \text{OPT}(j) \) = value of optimal solution to the problem consisting of job requests 1, 2, ..., \( j \).

- Case 1: \( \text{OPT} \) selects job \( j \).
  - can’t use incompatible jobs \( \{ p(j) + 1, p(j) + 2, ..., j - 1 \} \)
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., \( p(j) \)

- Case 2: \( \text{OPT} \) does not select job \( j \).
  - must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., \( j - 1 \)

\[
\text{OPT}(j) = \begin{cases} 
0 & \text{if } j = 0 \\
\max \{ v_j + \text{OPT}(p(j)), \text{OPT}(j - 1) \} & \text{otherwise}
\end{cases}
\]

Weighted Interval Scheduling: Brute Force

Brute force algorithm.

Input: \( n, s_1, ..., s_n, f_1, ..., f_n, v_1, ..., v_n \)
Sort jobs by finish times so that \( f_1 \leq f_2 \leq ... \leq f_n \).
Compute \( p(1), p(2), ..., p(n) \)

Compute-Opt(\( j \)) {
  if (\( j = 0 \))
    return 0
  else
    return \( \max(v_j + \text{Compute-Opt}(p(j)), \text{Compute-Opt}(j - 1)) \)
}

return \( \max(v_j + \text{Compute-Opt}(p(j)), \text{Compute-Opt}(j - 1)) \)

Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

Input: \( n, s_1, ..., s_n, f_1, ..., f_n, v_1, ..., v_n \)
Sort jobs by finish times so that \( f_1 \leq f_2 \leq ... \leq f_n \).
Compute \( p(1), p(2), ..., p(n) \)

M-Compute-Opt(\( j \)) {
  if (\( M[j] \) is empty)
    \( M[j] \) = empty — global array
    \( M[j] \) = 0
  \( M[j] \) = \( \max(v_j + M-\text{Compute-Opt}(p(j)), M-\text{Compute-Opt}(j - 1)) \)
  return \( M[j] \)
}

Memoization. Store results of each sub-problem in a cache; lookup as needed.
Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.
- Sort by finish time: $O(n \log n)$.
- Computing $p()$: $O(n)$ after sorting by start time.
- $M$-Compute-Opt: each invocation takes $O(1)$ time and either
  - (i) returns an existing value $M[j]$
  - (ii) fills in one new entry $M[j]$ and makes two recursive calls
- Progress measure $\Phi = \#$ nonempty entries of $M$:
  - initially $\Phi = 0$, throughout $\Phi \leq n$.
  - (ii) increases $\Phi$ by 1 $\Rightarrow$ at most $2n$ recursive calls.
- Overall running time of $M$-Compute-Opt is $O(n)$.

Remark. $O(n)$ if jobs are pre-sorted by start and finish times.

Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

Input: $n$, $s_1, \ldots, s_n$, $f_1, \ldots, f_n$, $v_1, \ldots, v_n$
Sort jobs by finish times so that $f_1 \leq f_2 \leq \ldots \leq f_n$.
Compute $p(1), p(2), \ldots, p(n)$
Iterative-Compute-Opt {
  $M[0] = 0$
  for $j = 1$ to $n$
    $M[j] = \max(v_j + M[p(j)], M[j-1])$
}

Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?
A. Do some post-processing.

Run $M$-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
  if ($j = 0$)
    output nothing
  else if ($v_j + M[p(j)] > M[j-1]$)
    print $j$
    Find-Solution(p(j))
  else
    Find-Solution(j-1)
}

- # of recursive calls $\leq n \Rightarrow O(n)$. 

Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \ldots \leq f_n$.
Def. $p(j)$ = largest index $i < j$ such that job $i$ is compatible with $j$.
Ex: $p(8) = 5, p(7) = 3, p(2) = 0$.