Integers II
CSE 410 Winter 2017

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Teaching Assistants: Kathryn Chan, Kevin Bi, Ryan Wong, Waylon Huang, Xinyu Sui
Administrivia

- Lab 1 due next Thursday (1/26)
  - Prelim submission (3+ of `bits.c`) due on Thursday (1/19)
  - Bonus slides at the end of today’s lecture have relevant examples
- Homework 2 will be released on Friday (1/20)
  - Timing overlaps a bit with Lab 1
- Optional section tomorrow @ 4:30pm in MGH 241
  - Integers and logical operators
Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations
Two’s Complement Arithmetic

- The same addition procedure works for both unsigned and two’s complement integers
  - **Simplifies hardware**: only one algorithm for addition
  - **Algorithm**: simple addition, **discard the highest carry bit**
    - Called modular addition: result is sum \( \text{modulo } 2^w \)

- 4-bit Examples:

<table>
<thead>
<tr>
<th></th>
<th>0100</th>
<th>1100</th>
<th>0100</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>+3</td>
<td>+4</td>
<td>+4</td>
</tr>
<tr>
<td>3</td>
<td>0011</td>
<td>0011</td>
<td>1101</td>
</tr>
<tr>
<td></td>
<td>=7</td>
<td>=-1</td>
<td>=1</td>
</tr>
</tbody>
</table>
Why Does Two’s Complement Work?

For all representable positive integers \( x \), we want:

\[
\text{bit representation of } x + \text{bit representation of } -x = 0 \quad \text{(ignoring the carry-out bit)}
\]

What are the 8-bit negative encodings for the following?

\[
\begin{align*}
00000001 & + ????????? \quad 00000010 & + ????????? \quad 11000011 & + ????????? \\
00000000 & \quad 00000000 & \quad 00000000 & \quad 00000000
\end{align*}
\]
Why Does Two’s Complement Work?

- For all representable positive integers $x$, we want:
  \[
  \text{bit representation of } x + \text{bit representation of } -x = 0 \quad \text{(ignoring the carry-out bit)}
  \]

- What are the 8-bit negative encodings for the following?

  \[
  \begin{align*}
  00000001 & \quad + \quad 11111111 & \quad = \quad 100000000 \\
  00000010 & \quad + \quad 11111110 & \quad = \quad 100000000 \\
  11000011 & \quad + \quad 00111101 & \quad = \quad 100000000 \\
  \end{align*}
  \]

  These are the bitwise complement plus 1!

  \[
  -x \equiv \sim x + 1
  \]
Signed/Unsigned Conversion Visualized

- Two’s Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive
Values To Remember

- **Unsigned Values**
  - $U_{\text{Min}} = 0b00...0 = 0$
  - $U_{\text{Max}} = 0b11...1 = 2^w - 1$

- **Two’s Complement Values**
  - $T_{\text{Min}} = 0b10...0 = -2^{w-1}$
  - $T_{\text{Max}} = 0b01...1 = 2^{w-1} - 1$
  - $-1 = 0b11...1$

- **Example:** Values for $w = 64$

<table>
<thead>
<tr>
<th></th>
<th>Decimal</th>
<th>Hex</th>
</tr>
</thead>
<tbody>
<tr>
<td>UMax</td>
<td>18,446,744,073,709,551,615</td>
<td>FF FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>Tmax</td>
<td>9,223,372,036,854,775,807</td>
<td>7F FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>Tmin</td>
<td>-9,223,372,036,854,775,808</td>
<td>80 00 00 00 00 00 00 00</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>FF FF FF FF FF FF FF FF FF</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>00 00 00 00 00 00 00 00</td>
</tr>
</tbody>
</table>

Decimal to Hex conversion:
- $18,446,744,073,709,551,615$ in decimal is $FF FF FF FF FF FF FF FF FF$ in hex.
- $9,223,372,036,854,775,807$ in decimal is $7F FF FF FF FF FF FF FF FF$ in hex.
- $-9,223,372,036,854,775,808$ in decimal is $80 00 00 00 00 00 00 00$ in hex.
- $-1$ in decimal is $FF FF FF FF FF FF FF FF FF$ in hex.
- $0$ in decimal is $00 00 00 00 00 00 00 00$ in hex.
In C: Signed vs. Unsigned

- **Casting**
  - Bits are unchanged, just interpreted differently!
    - `int` tx, ty;
    - `unsigned int` ux, uy;
  - *Explicit* casting
    - tx = `(int)` ux;
    - uy = `(unsigned int)` ty;
  - *Implicit* casting can occur during assignments or function calls
    - tx = ux;
    - uy = ty;
Casting Surprises

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use “U” (or “u”) suffix to explicitly force *unsigned*
    - Examples: 0U, 4294967259u

- Expression Evaluation
  - When you mixed unsigned and signed in a single expression, then *signed values are implicitly cast to unsigned*
  - Including comparison operators <, >, ==, <=, >=
Casting Surprises

- 32-bit examples:
  - Tmin = -2,147,483,648, Tmax = 2,147,483,647

<table>
<thead>
<tr>
<th>Left Constant</th>
<th>Order</th>
<th>Right Constant</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td>0U</td>
<td></td>
</tr>
<tr>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>0U</td>
<td></td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>0000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>-2147483648</td>
<td></td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
</tr>
<tr>
<td>2147483647U</td>
<td></td>
<td>-2147483648</td>
<td></td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td></td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
</tr>
<tr>
<td>(unsigned) -1</td>
<td></td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>1111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>2147483648U</td>
<td></td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
</tr>
<tr>
<td>2147483647</td>
<td></td>
<td>(int) 2147483648U</td>
<td></td>
</tr>
<tr>
<td>0111 1111 1111 1111 1111 1111 1111 1111</td>
<td></td>
<td>1000 0000 0000 0000 0000 0000 0000 0000</td>
<td></td>
</tr>
</tbody>
</table>
Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations
Arithmetic Overflow

- When a calculation produces a result that can’t be represented in the current encoding scheme
  - Integer range limited by fixed width
  - Can occur in both the positive and negative directions

- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication... oops!

<table>
<thead>
<tr>
<th>Bits</th>
<th>Unsigned</th>
<th>Signed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0001</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0010</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>0011</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>0100</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>0101</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>0110</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>0111</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>1000</td>
<td>8</td>
<td>-8</td>
</tr>
<tr>
<td>1001</td>
<td>9</td>
<td>-7</td>
</tr>
<tr>
<td>1010</td>
<td>10</td>
<td>-6</td>
</tr>
<tr>
<td>1011</td>
<td>11</td>
<td>-5</td>
</tr>
<tr>
<td>1100</td>
<td>12</td>
<td>-4</td>
</tr>
<tr>
<td>1101</td>
<td>13</td>
<td>-3</td>
</tr>
<tr>
<td>1110</td>
<td>14</td>
<td>-2</td>
</tr>
<tr>
<td>1111</td>
<td>15</td>
<td>-1</td>
</tr>
</tbody>
</table>
Overflow: Unsigned

- **Addition:** drop carry bit ($-2^N$)
  
  \[
  \begin{array}{c@{\quad}c@{\quad}c@{\quad}c@{\quad}c}
  15 & 1111 \\
  + & 0010 & \rightarrow & 10001 \\
  \hline
  17 & 1 & \underline{0} & 1 & 1
  \end{array}
  \]

- **Subtraction:** borrow ($+2^N$)
  
  \[
  \begin{array}{c@{\quad}c@{\quad}c@{\quad}c@{\quad}c}
  1 & 10001 \\
  - & 0010 & \rightarrow & 11111 \\
  \hline
  15 & 1 & \underline{1} & 1 & 1
  \end{array}
  \]

$\pm2^N$ because of modular arithmetic
Overflow: Two’s Complement

- Addition: \((+)+(+)=(-)\) result?

\[
\begin{array}{c}
6 \\
+ \ 3 \\
\hline
9 \\
-7
\end{array}
\quad 0110 \\
\begin{array}{c}
6 \\
+ 0011 \\
\hline
1001
\end{array}
\]

- Subtraction: \((-)+(-)=(+)?

\[
\begin{array}{c}
-7 \\
- 3 \\
\hline
-10 \\
6
\end{array}
\quad 1001 \\
\begin{array}{c}
-7 \\
- 0011 \\
\hline
0110
\end{array}
\]

For signed: overflow if operands have same sign and result’s sign is different
Sign Extension

- What happens if you convert a *signed* integral data type to a larger one?
  - *e.g.* char → short → int → long

- **4-bit → 8-bit Example:**
  - Positive Case
    - 4-bit: 0010 = +2
    - 8-bit: 00000010 = +2
Peer Instruction Question

- Which of the following 8-bit numbers has the same signed value as the 4-bit number 0b1100?
  - Underlined digit = MSB
  - Vote at http://PollEv.com/justinh

A. 0b 0000 1100
B. 0b 1000 1100
C. 0b 1111 1100
D. 0b 1100 1100
E. We’re lost...
Sign Extension

- **Task:** Given a $w$-bit signed integer $X$, convert it to $w+k$-bit signed integer $X'$ *with the same value*

- **Rule:** Add $k$ copies of sign bit
  - Let $x_i$ be the $i$-th digit of $X$ in binary
  - $X' = x_{w-1}, \ldots, x_{w-1}, x_{w-1}, x_{w-2}, \ldots, x_1, x_0$

\[\begin{align*}
\text{original } X & \quad \text{\(w\) copies of MSB} \\
X' & \quad k \text{ copies of MSB}
\end{align*}\]
Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
  - Java too

```java
short int x = 12345;
int ix = (int) x;
short int y = -12345;
int iy = (int) y;
```

<table>
<thead>
<tr>
<th>Var</th>
<th>Decimal</th>
<th>Hex</th>
<th>Binary</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>12345</td>
<td>30 39</td>
<td>00110000 00111001</td>
</tr>
<tr>
<td>ix</td>
<td>12345</td>
<td>00 00 30 39</td>
<td>00000000 00000000 00110000 00111001</td>
</tr>
<tr>
<td>y</td>
<td>-12345</td>
<td>CF C7</td>
<td>11001111 11000111</td>
</tr>
<tr>
<td>iy</td>
<td>-12345</td>
<td>FF FF CF C7</td>
<td>11111111 11111111 11001111 11000111</td>
</tr>
</tbody>
</table>
Integers

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Shift Operations

- Left shift \((x << n)\) bit vector \(x\) by \(n\) positions
  - Throw away (drop) extra bits on left
  - Fill with 0s on right

- Right shift \((x >> n)\) bit-vector \(x\) by \(n\) positions
  - Throw away (drop) extra bits on right
  - Logical shift (for unsigned values)
    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
    - Maintains sign of \(x\)
Shift Operations

- **Left shift** $(x<<n)$
  - Fill with 0s on right

- **Right shift** $(x>>n)$
  - **Logical shift** (for **unsigned** values)
    - Fill with 0s on left
  - **Arithmetic shift** (for **signed** values)
    - Replicate most significant bit on left

**Notes:**
- Shifts by $n<0$ or $n \geq w$ (bit width of $x$) are **undefined**
- **In C:** behavior of $>>$ is determined by compiler
  - In gcc / C lang, depends on data type of $x$ (signed/unsigned)
- **In Java:** logical shift is $>>>$ and arithmetic shift is $>>$
Shifting Arithmetic?

- What are the following computing?
  - \( x \gg n \)
    - \( 0b\ 0100 \gg 1 = 0b\ 0010 \)
    - \( 0b\ 0100 \gg 2 = 0b\ 0001 \)
    - **Divide** by \( 2^n \)
  - \( x \ll n \)
    - \( 0b\ 0001 \ll 1 = 0b\ 0010 \)
    - \( 0b\ 0001 \ll 2 = 0b\ 0100 \)
    - **Multiply** by \( 2^n \)

- Shifting is faster than general multiply and divide operations
Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
- Difference comes during interpretation: $x \times 2^n$?

\[
\begin{align*}
  x & = 25; \quad 00011001 = 25 & \text{Signed} & 25 & \text{Unsigned} & 25 \\
  L1 = x \ll 2; & \quad 0001100100 = 100 & \text{Signed} & 100 & \text{Unsigned} & 100 \\
  L2 = x \ll 3; & \quad 00011001000 = -56 & \text{Signed} & 112 & \text{Unsigned} & 144 \\
  L3 = x \ll 4; & \quad 000110010000 = -112 & \text{Signed} & 112 & \text{Unsigned} & 144
\end{align*}
\]
Right Shifting Arithmetic 8-bit Examples

- **Reminder**: C operator `>>` does logical shift on unsigned values and arithmetic shift on signed values
  - Logical Shift: \( x / 2^n \)?

\[
x_u = 240u; \quad 11110000 = 240
\]

\[
R_{1u} = x_u >> 3; \quad 00011110000 = 30
\]

\[
R_{2u} = x_u >> 5; \quad 0000011110000 = 7
\]
Right Shifting Arithmetic 8-bit Examples

Reminder: C operator >> does \textit{logical} shift on unsigned values and \textit{arithmetic} shift on signed values

- Arithmetic Shift: \(x/2^n\)?

\[
\begin{align*}
x_s &= -16; & 11110000 &= -16 \\
R1_s &= xu >> 3; & 11111110000 &= -2 \\
R2_s &= xu >> 5; & 111111110000 &= -1
\end{align*}
\]
Peer Instruction Question

For the following expressions, find a value of signed char \( x \), if there exists one, that makes the expression TRUE. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:
  - \( x == (\text{unsigned char}) x \)
  - \( x >= 128U \)
  - \( x != (x>>2) \ll 2 \)
  - \( x == -x \)
    - Hint: there are two solutions
  - \( (x < 128U) \&\& (x > 0\times3F) \)
Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just interpreted differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)

- We can only represent so many numbers in \( w \) bits
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding

- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking
Summary

http://xkcd.com/571/
Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1. We will try to cover these in lecture or section if we have the time.

- Extract the 2^{nd} most significant byte of an \texttt{int}
- Extract the sign bit of a signed \texttt{int}
- Conditionals as Boolean expressions
Using Shifts and Masks

- Extract the 2\textsuperscript{nd} most significant \textit{byte} of an \texttt{int}:
  - First shift, then mask: \((x\gg16) \& 0xFF\)
  - Or first mask, then shift: \((x \& 0xFF0000) \gg 16\)

<table>
<thead>
<tr>
<th>x</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>x\gg16</td>
<td>00000000 00000000 00000011 00000100</td>
</tr>
<tr>
<td>0xFF</td>
<td>00000000 00000000 00000000 11111111</td>
</tr>
<tr>
<td>(x\gg16) &amp; 0xFF</td>
<td>00000000 00000000 00000000 00000100</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>0xFF0000</td>
<td>00000000 11111111 00000000 00000000</td>
</tr>
<tr>
<td>x &amp; 0xFF0000</td>
<td>00000000 00000010 00000000 00000000</td>
</tr>
<tr>
<td>(x&amp;0xFF0000) \gg 16</td>
<td>00000000 00000000 00000000 00000100</td>
</tr>
</tbody>
</table>
Using Shifts and Masks

- **Extract the *sign bit* of a signed int:**
  - **First shift, then mask:** \((x >> 31) \& \ 0x1\)
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear 1s possibly shifted in

<table>
<thead>
<tr>
<th>x</th>
<th>00000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>x&gt;&gt;31</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td>0x1</td>
<td>00000000 00000000 00000000 00000001</td>
</tr>
<tr>
<td>(x&gt;&gt;31) &amp; 0x1</td>
<td>00000000 00000000 00000000 00000000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>10000001 00000010 00000011 00000100</th>
</tr>
</thead>
<tbody>
<tr>
<td>x&gt;&gt;31</td>
<td>11111111 11111111 11111111 11111111</td>
</tr>
<tr>
<td>0x1</td>
<td>00000000 00000000 00000000 00000001</td>
</tr>
<tr>
<td>(x&gt;&gt;31) &amp; 0x1</td>
<td>00000000 00000000 00000000 00000001</td>
</tr>
</tbody>
</table>
Using Shifts and Masks

- **Conditionals as Boolean expressions**
  - For `int x`, what does `(x<<31)>>31` do?

<table>
<thead>
<tr>
<th></th>
<th>00000000 00000000 00000000 00000001</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>x=!123</code></td>
<td>[269x133] 10000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td><code>x&lt;&lt;31</code></td>
<td>11111111111111111111111111111111</td>
</tr>
<tr>
<td><code>(x&lt;&lt;31)&gt;&gt;31</code></td>
<td>[237x120] 00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td><code>!x</code></td>
<td>[366x127] 00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td><code>!x&lt;&lt;31</code></td>
<td>[398x87] 00000000 00000000 00000000 00000000</td>
</tr>
<tr>
<td><code>(!!x&lt;&lt;31)&gt;&gt;31</code></td>
<td>[465x74] 00000000 00000000 00000000 00000000</td>
</tr>
</tbody>
</table>

- **Can use in place of conditional:**
  - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
  - `a=((x<<31)>>31)&y) | ((!!x<<31)>>31)&z);`