Decimal and binary representation systems

• They both are *positional* representation systems
• In decimal, numbers are represented by the coefficients of the powers of 10
  – Example: $321 = 3 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0$
• Decomposing 321 in powers of 2 yields $321 = 256 + 64 + 1$
  – or $1.2^8 + 0.2^7 + 1.2^6 + 0.2^5 + 0.2^4 + 0.2^3 + 0.2^2 + 0.2^1 + 1.2^0$
  – or $101000001_2$
Positional number systems

- More generally, a positive integer is represented in
  - decimal by $\sum_{i=0}^{n} a_i \times 10^{n-i}$ where $0 \leq a_i \leq 9$
  - binary by $\sum_{i=0}^{m} b_i \times 2^{m-i}$ where $0 \leq b_i \leq 1$
Binary and Hexadecimal representation systems

- Writing binary numbers quickly becomes error-prone and unwieldy
- Instead use hexadecimal system, positional representation system in base 16
  - Example: $321 = 1 \cdot 16^2 + 4 \cdot 16^1 + 1 \cdot 16^0 = 141_{16}$
- Since the coefficients are between 0 and 15, we need new symbols to represent 10 through 15. They will be A through F
  - A (hex) = 10 (decimal) = 1010 (binary)
  - B (hex) = 11 (decimal) = 1011 (binary) ...
  - F (hex) = 15 (decimal) = 1111 (binary)
Conversion between binary and hexadecimal

• Group *bits* (abbreviation for binary digits) by groups of four, starting from the right (*least significant bit* or *lsb*)
  – Example: $101000001 = 10100001$ (binary) = 141 (hex)
  – $111001011 = 111001011$ (binary) = 1CB (hex)
  – Note that the greatest magnitude bit, the leftmost one, is called (*most significant bit* or *msb*)

• Why hexadecimal?
  – Very convenient to represent strings of 4, 8, …, 16, …32, …64 bits by 1, 2, …4, …8, …16 hex digits
Some useful powers of two

\[ 2^{10} = 1024_{10} \approx 10^3 = 1K(Kilo) \]
\[ 2^{20} \approx 10^6 = 1M(Mega) \]
\[ 2^{30} \approx 10^9 = 1G(Giga) \]

- We’ll often round-off and talk about, say, 16 KB or 64 MB
Representing positive and negative integers

• In an $n$-bit register, you can represent $2^n$ patterns
  – Example: in a 32-bit register, we can represent *unsigned integers*
    in the range $[0:2^{32}-1]$

• How to represent positive and negative integers with the following properties:
  – Equal number of positive and negative numbers
  – Unique (and easily testable) representation of zero
  – Easy sign test
  – Easy rules for addition and subtraction
Three representation systems

• Historically, 3 different numbering systems have been used
  – Two’s complement now used in all machines for integer representation
  – sign and magnitude used (partially) for floating-point representation
  – One’s complement (very similar to 2’s complement but with a few more drawbacks)
Two’s complement representation

- Positive numbers as unsigned binary with msb always 0
- Zero is represented as a string of 0’s
- To represent a negative number:
  - Consider the representation of its absolute value
  - Flip all 1’s to 0’s and 0’s to 1’s (this is 1’s complement)
  - Add 1 to lsb using binary arithmetic rules
2’s complement

- Example assuming a 4-bit register
  - What is the representation of (decimal) 6?
  - What is the representation of (decimal) -6?
  - What is the representation of 0?
  - What is the range of representation of positive numbers?
  - What is the range of representation of negative numbers?
  - How do I recognize whether a number is positive or negative or zero?
Addition in 2’s complement

- **Addition**
  - Perform an ordinary binary addition and discard the carry-out.
  - If you add 2 numbers of opposite sign, everything will always be all right.
  - If you add two positive numbers and the result *appears* to be negative (i.e., $\text{msb} = 1$) then you have an *overflow*. This will generate an *exception* in your program.
  - If you add two negative numbers and the result *appears* to be positive (i.e., $\text{msb} = 0$) then you have an *underflow*.

- **Subtraction**
  - Take the 2’s complement of the subtrahend and add it to the other operand.