Lecture 3: How to measure efficiency

Data Structures and Algorithms

Announcements

- Course background survey due by Friday

- HW 1 is Due Friday

- Alex has Office Hours after class (2:30-4:30) CSE 006, will help with setup
  - If you have any questions about your setup please come to office hours so we can iron out all the wrinkles before the partnered projects begin next week.

- HW 2 Assigned on Friday – Partner selection forms due by 11:59pm **Thursday**

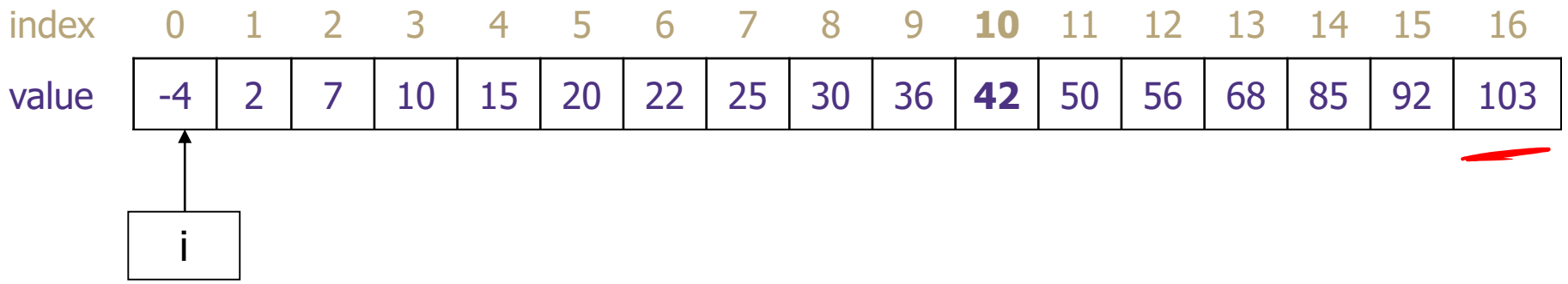
https://goo.gl/forms/rVrVUkFDdsqI8pkD2

Review: Sequential Search

sequential search: Locates a target value in an array / list by examining each element from start to finish.

- How many elements will it need to examine?

- Example: Searching the array below for the value **42**:

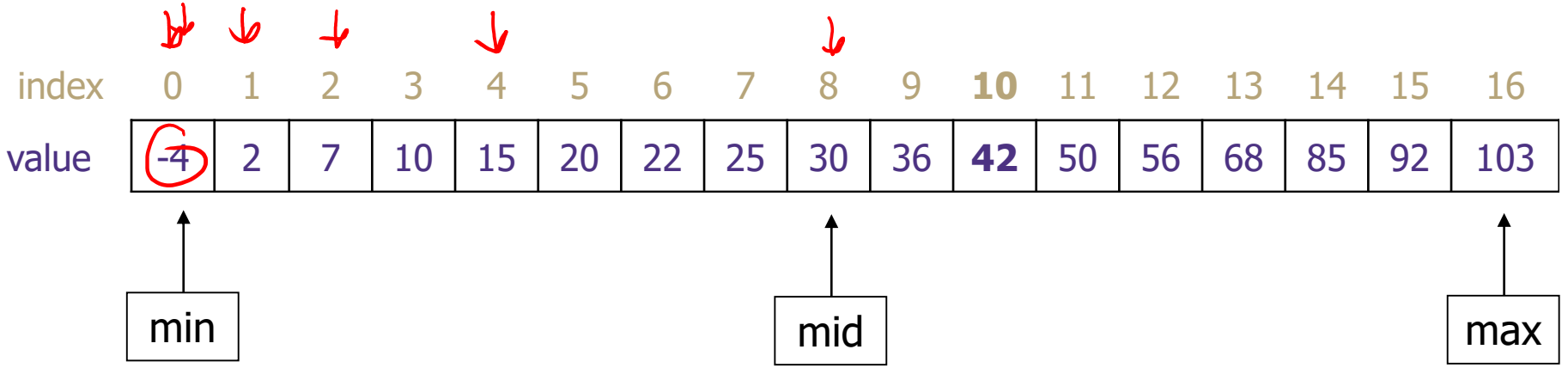


- What is the best case? O(1)
- What is the worst case? O(n)
- What is the complexity class? O(n)

Review: Binary Search

binary search: Locates a target value in a *sorted* array or list by successively eliminating half of the array from consideration.

- How many elements will it need to examine?
- Example: Searching the array below for the value **42**:



- What is the best case? O(1) - the middle
- What is the worst case? The beginning
- What is the complexity class? log(n)

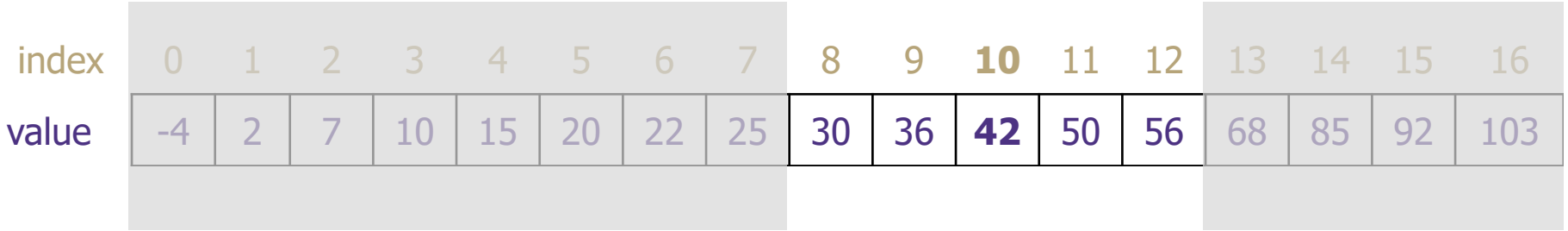
Analyzing Binary Search

What is the pattern?

- At each iteration, we eliminate half of the remaining elements

How long does it take to finish?

- 1st iteration – N/2 elements remain
- 2nd iteration – N/4 elements remain
- Kth iteration - N/2^k elements remain
- Done when N/2^k = 1



Analyzing Binary Search

$$\frac{N}{2^K} = 1$$

$$N = 2^K$$

$$\log_2 N = \log_2 2^K$$

$$\boxed{\log_2 N = K}$$

$O(\log n)$

↑ we lost our base!

Logarithms

$\log_b a = x$ mean

x solves

$b^? = a$

$\log_b b^z = x$

$b^? = b^z \rightarrow z$

Analyzing Binary Search

Finishes when $N / 2^K = 1$

$N / 2^K = 1$

-> multiply right side by 2^K

$N = 2^K$

-> isolate K exponent with logarithm

$Log_2 N = k$

Is this exact?

- N can be things other than powers of 2
- If N is odd we can't technically use Log_2
- When we have an odd number of elements we select the larger half
- Within a fair rounding error

Asymptotic Analysis

asymptotic analysis: how the runtime of an algorithm grows as the data set grows

Approximations / Rules

- Basic operations take "constant" time
  - Assigning a variable
  - Accessing a field or array index
- Consecutive statements
  - Sum of time for each statement
- Function calls
  - Time of function's body
- Conditionals
  - Time of condition + maximum time of branch code
- Loops
  - Number of iterations x time for loop body

Modeling Case Study

Goal: return 'true' if a sorted array of ints contains duplicates

Solution 1: compare each pair of elements

```
public boolean hasDuplicate1(int[] array) {
   for (int i = 0; i < array.length; i++) {
      for (int j = 0; j < array.length; j++) {
         if (i != j && array[i] == array[j]) {
            return true;
         }
      }
   }
   return false;
}
```

Solution 2: compare each consecutive pair of elements

```
public boolean hasDuplicate2(int[] array) {
   for (int i = 0; i < array.length - 1; i++) {
      if (array[i] == array[i + 1]) {
            return true;
      }
   }
   return false;
}
```

Modeling Case Study: Solution 2

T(n) where n = array.length

-> work inside out

Solution 2: compare each consecutive pair of elements

```
public boolean hasDuplicate2(int[] array) {
    for (int i = 0; i < array.length - 1; i++) {
        if (array[i] == array[i + 1]) { +4
            return true; +1
        }
    }
    return false; +1
}
```

If statement = 5

x n - 1

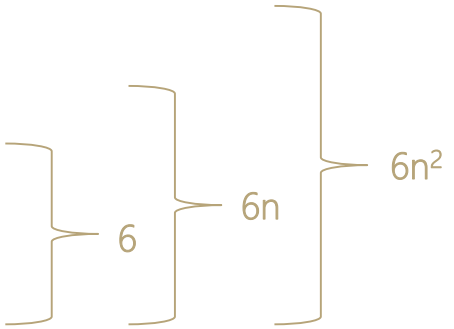
T(n) = 5 (n-1) + 1

linear time complexity class O(n)

Modeling Case Study: Solution 1

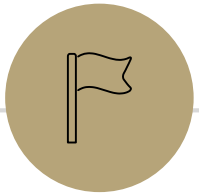
Solution 1: compare each consecutive pair of elements

```
public boolean hasDuplicate1(int[] array) {
    for (int i = 0; i < array.length; i++) {          x n
        for (int j = 0; j < array.length; j++) {      x n
            if (i != j && array[i] == array[j]) {     +5
                return true;                          +1
            }
        }
    }
    return false;                                     +1
}
```



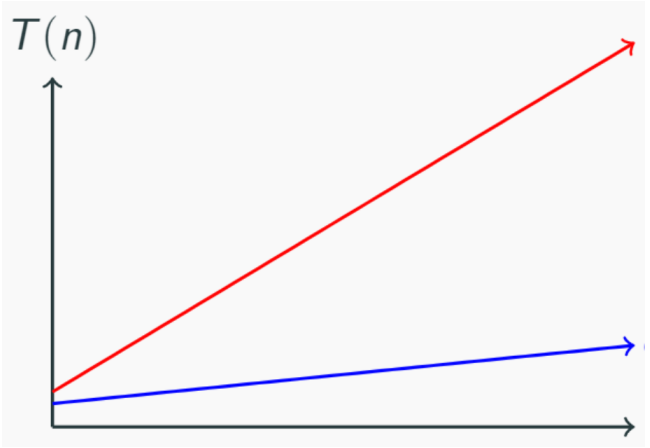
$T(n) = 6\ n^2 + 1$

quadratic time complexity class $O(n^2)$

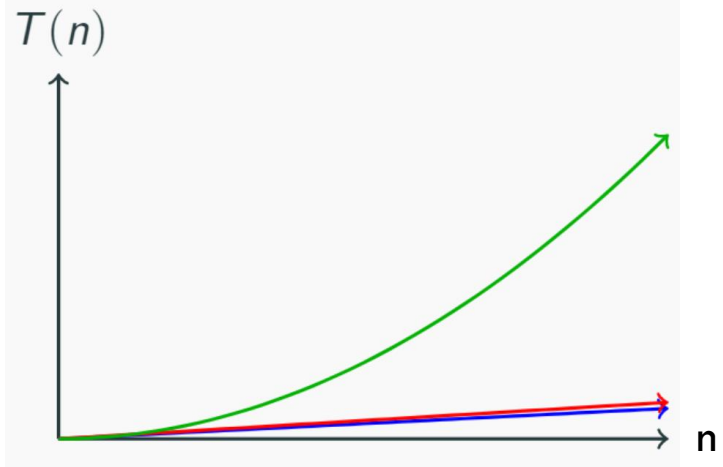


Comparing Functions

Function growth

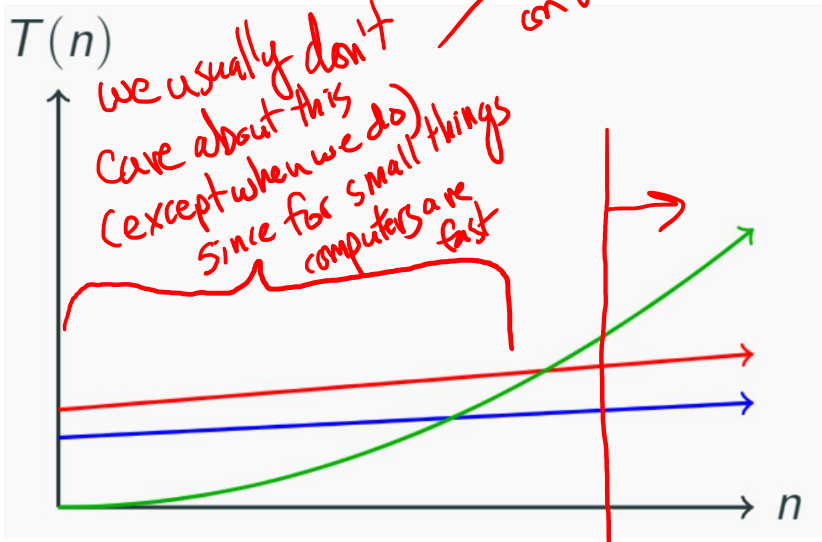


n and 4n look very different up close



n and 4n look the same over time

n^2 eventually dominates n



n^2 doesn't start off dominating the linear functions

It eventually takes over…

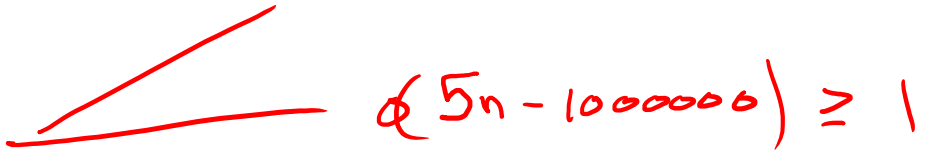
Function comparison: exercise

$O(n) = O(5n+3)$ $O(n) \leq O(5n+3)$

$f(n) = n \leq g(n) = 5n + 3$? **True** – all linear functions are treated as equivalent

$f(n) = 5n + 3 \leq g(n) = n$? **True**

$f(n) = 5n + 3 \leq g(n) = 1$? **False**



$f(n) = 5n + 3 \leq g(n) = n^2$? **True** – quadratic will always dominate linear

$f(n) = n^2 + 3n + 2 \leq g(n) = n^3$? **True**

$f(n) = n^3 \leq g(n) = n^2 + 3n + 2$? **False**

Definition: function domination

Definition: Domination

A function f(n) is dominated by g(n) when...

There exists two constants c > 0 and $n_0 > 0$

Such that for all values of $n \geq n_0$

$f(n) \leq c * g(n)$

Example:

Is f(n) = n dominated by g(n) = 5n + 3 ?

c = 1

n_0 = 1

Yes!

$2n = n+3$

$n_0 = 10$

$f(n) = n$

$g(n) = n+3$

$n_0 \geq 1$

$c = 1$

$c \cdot g(n) = n+3 \geq n \quad \forall n > n_0 = 1$

$c = 2$

$c \cdot f(n) = 2n \geq n+3 \quad \forall n > 10$

Exercise: Function Domination

Demonstrate that $5n^2 + 3n + 6$ is dominated by n^3 by finding a c and n_0 that satisfy the definition of domination

$5n^2 + 3n + 6 \leq 5n^2 + 3n^2 + 6n^2$ when $n \geq 1$

$5n^2 + 3n^2 + 6n^2 = 14n^2$

$5n^2 + 3n + 6 \leq 14n^2$ for $n \geq 1$

$14n^2 \leq c*n^3$ for c = ? n >= ?

$\frac{14}{n}$ -> c = 14 & n >= 1

n=1 LHS 5+3+6 = 14

5+3+6 = 14

Definition: Big O

If $f(n) = n \leq g(n) = 5n + 3 \leq h(n) = 100n$ and

$h(n) = 100n \leq g(n) = 5n + 3 \leq f(n)$ n

Really they are all the "same"

Definition: Big O

$O(f(n))$ is the "family" or "set" of all functions that are dominated by f(n)

Question: are O(n), O(5n + 3) and O(100n) all the same?

True! By convention we pick simplest of the above -> O(n) ie "linear"

f(n) = 1 in O(n)?

Definitions: Big Ω

"f(n) is greater than or equal to g(n)"

F(n) dominates g(n) when:

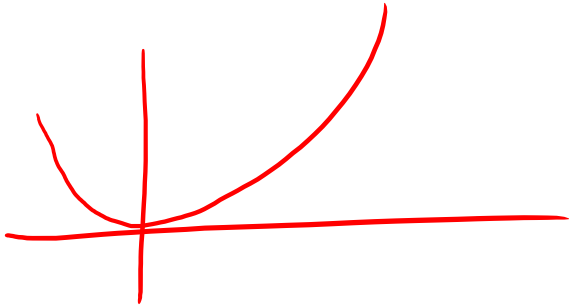
There exists two constants such that c > 0 and n0 > 0

Such that for all values n >= n0

F(n) >= c * g(n) is true

Definition: Big Ω

Ω(f(n)) is the family of all functions that dominates f(n)



n^2

$\Omega(n^2)$

$\rightarrow n^3$

$\rightarrow n^5$

$\rightarrow n!$

∈ Element Of

f(n) is dominated by g(n)

Is that the same as

"f(n) is contained inside O(g(n))"

Yes!

f(n) ∈ g(n)

$n = O(n)$

$n \leq O(n)$

$n \in O(n)$

Examples

$4n^2 \in \Omega(1)$

true

$4n^2 \in \Omega(n)$

true

$4n^2 \in \Omega(n^2)$

true

$4n^2 \in \Omega(n^3)$

false

$4n^2 \in \Omega(n^4)$

false

$4n^2 \in O(1)$

false

$4n^2 \in O(n)$

false

$4n^2 \in O(n^2)$

true

$4n^2 \in O(n^3)$

true

$4n^2 \in O(n^4)$

true

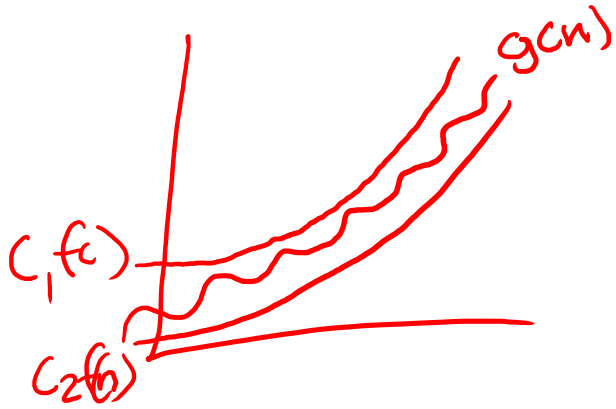
Definition: Big O

***O*(f(n))** is the "family" or "set" of all functions that are dominated by f(n)

Definition: Big Ω

Ω(f(n)) is the family of all functions that dominates f(n)

Definitions: Big Θ



We say f(n) ∈ Θ(g(n)) when both

f(n) ∈ O(g(n)) and f(n) ∈ Ω (g(n)) are true

Which is only when f(n) = g(n)

Definition: Big Θ

Θ(f(n)) is the family of functions that are equivalent to f(n)

Industry uses "Big Θ" and "Big O" interchangeably

Summary

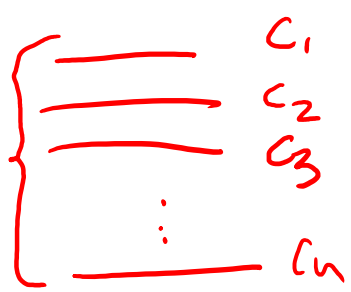
$O(f(n)) \leq f(n) == \Theta(f(n)) \leq \Omega(f(n))$

f(n)	Dominated by $f(n) \in O(g(n))$	Dominates $f(n) \in \Omega(g(n))$
O(1)		
O(log n)		
O(n)		
$O(n^2)$		
$O(n^3)$		

Justifying the "Rules"

Approximations / Rules

- Basic operations take "constant" time
  - Assigning a variable
  - Accessing a field or array index
- Consecutive statements
  - Sum of time for each statement
- Function calls
  - Time of function's body
- Conditionals
  - Time of condition + maximum time of branch code
- Loops
  - Number of iterations x time for loop body



$c_1 + c_2 + \cdots + c_n \leq c \cdot n$

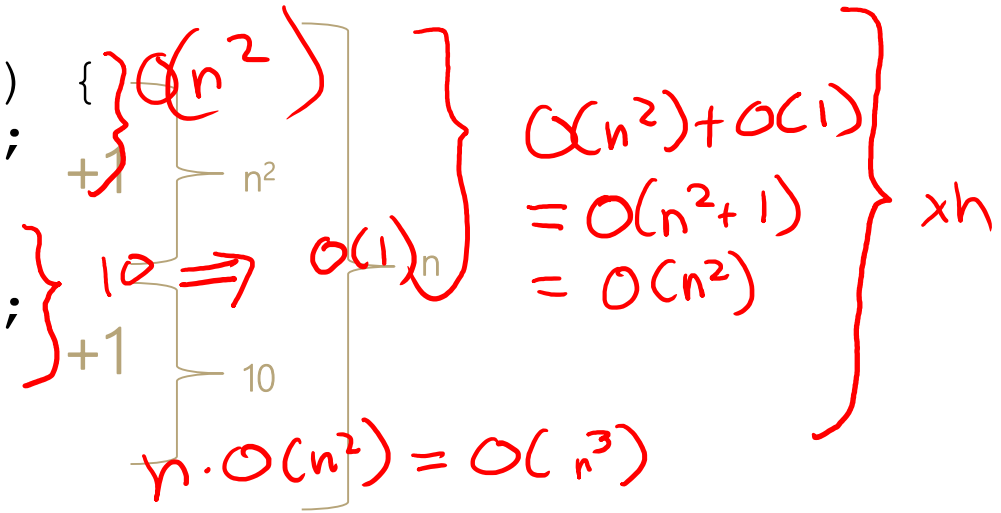
$n \uparrow \max c_n$

+1 per line

$c + 1 \leq c'$

A Slightly Harder example

```
public void mystery(int n) {
    for (int i = 0; i < n; i++) {
        for (int j = 0; j < n * n; j++) {
            System.out.println("Hello");
        }
        for (int j = 0; j < 10; j++) {
            System.out.println("world");
        }
    }
}
```

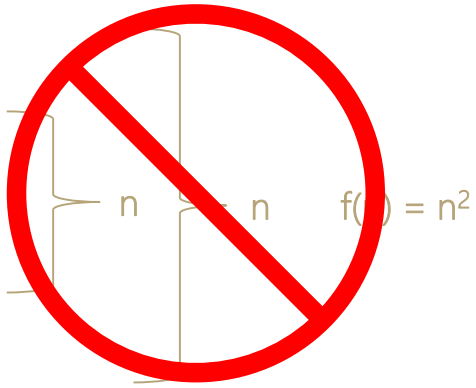


Remember: work outside in

Solution: $T(n) = n(n^2 + 10) = n^3 + 10n$

Modeling Complex Loops

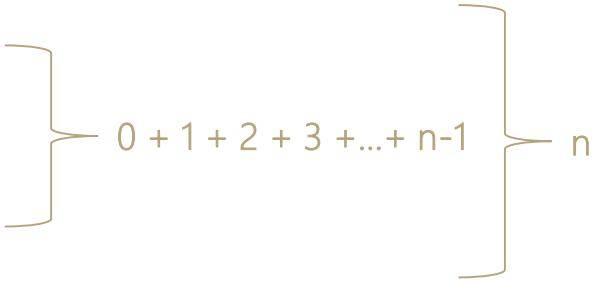
```
for (int i = 0; i < n; i++) {
    for (int j = 0; j < i; j++) {
        System.out.println("Hello!");
    }
}
```



Keep an eye on loop bounds!

Modeling Complex Loops

```
for (int i = 0; i < n; i++) {
    for (int j = 0; j < i; j++) {
        System.out.println("Hello!"); +c
    }
}
```



Summation
$1 + 2 + 3 + 4 + \ldots + n = \sum_{i=1}^{n} i$

Definition: Summation

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + f(a+2) + \ldots + f(b-2) + f(b-1) + f(b)$$

$$T(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} c$$

Simplifying Summations

```
for (int i = 0; i < n; i++) {
   for (int j = 0; j < i; j++) {
      System.out.println("Hello!");
   }
}
```

+c } 0 + 1 + 2 + 3 +…+ n-1 } n

(0c + 1c + 2c + 3c + … + i-1c)
+ (0c + 1c + 2c + 3c + … + i-1c)
+ (0c + 1c + 2c + 3c + … + i-1c)
+ repeat n times

T(n) = $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} c$ = $\sum_{i=0}^{n-1} ci$ Summation of a constant

= $c \sum_{i=0}^{n-1} i$ Factoring out a constant

= $c \frac{n(n-1)}{2}$ Gauss's Identity

= $\frac{c}{2} n^2 - \frac{c}{2} n$ $O(n^2)$

Function Modeling: Recursion

```
public int factorial(int n) {
    if (n == 0 || n == 1) {  +3
        return 1; +1                    } +c
    } else {
        return n * factorial(n - 1);  +????
}
```

Function Modeling: Recursion

```
public int factorial(int n) {
    if (n == 0 || n == 1) {
        return 1;
    } else {
        return n * factorial(n - 1);
}
```

+c_1

+T(n-1)

+c_2

$$T(n) = \begin{cases} C_1 & \text{when n = 0 or 1} \\ C_2 + T(n-1) & \text{otherwise} \end{cases}$$

Definition: Recurrence

Mathematical equivalent of an if/else statement
f(n) =

Unfolding Method

$$T(n) = \begin{cases} C_1 & \text{when } n = 0 \text{ or } 1 \\ C_2 + T(n-1) & \text{otherwise} \end{cases}$$

$T(3) = C_2 + T(3-1) = C_2 + (C_2 + T(2-1)) = C_2 + (C_2 + (C_1)) = 2C_2 + C_1$

$$T(n) = C_1 + \sum_{i=0}^{n-1} C_2$$

Summation of a constant

$T(n) = C_1 + (n-1)C_2$