CSE373: Data Structures and Algorithms

Comparison Sorting II

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This lecture material represents the work of multiple instructors at the University of Washington. Thank you to all who have contributed!

The comparison sorting problem

Assume we have *n* comparable elements in an array and we want to rearrange them to be in increasing order

Input:
- An array `A` of data records
- A key value in each data record
- A comparison function (consistent and total)

Effect:
- Reorganize the elements of `A` such that for any `i` and `j`, if `i < j` then `A[i] ≤ A[j]`
- (Also, `A` must have exactly the same data it started with)
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort

Sorting: The Big Picture

Surprising amount of neat stuff to say about sorting:

Simple algorithms: O(n^2)	**Fancier algorithms: O($n \log n$)**	**Comparison lower bound: $\Omega(n \log n)$**	**Specialized algorithms: O(n)**	**Handling huge data sets**
Insertion sort **Selection sort** Shell sort …	**Heap sort** **Merge sort** **Quick sort** …		**Bucket sort** **Radix sort**	External sorting

Divide and conquer

Very important technique in algorithm design

1. Divide problem into smaller parts
2. Independently solve the simpler parts
   - Think recursion
   - Or potential parallelism
3. Combine solution of parts to produce overall solution

Divide-and-Conquer Sorting

Two great sorting methods are fundamentally divide-and-conquer

1. Merge sort: Sort the left half of the elements (recursively)
   Sort the right half of the elements (recursively)
   Merge the two sorted halves into a sorted whole
2. Quick sort: Pick a "pivot" element
   Divide elements into less-than pivot
   and greater-than pivot
   Sort the two divisions (recursively on each)
   Answer is sorted-less-than then pivot then
   sorted-greater-than

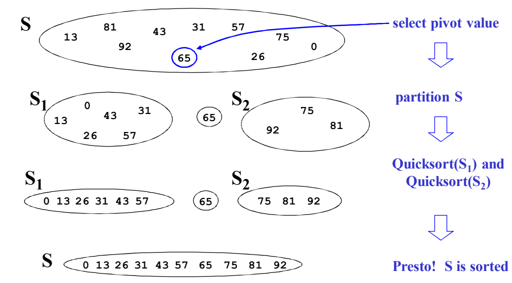
Quick sort

- A divide-and-conquer algorithm
  - Recursively chop into two pieces
  - Instead of doing all the work as we merge together, we will do all the work as we recursively split into halves
  - Unlike merge sort, does not need auxiliary space
- $O(n \log n)$ on average ☺, but $O(n^2)$ worst-case ☹
- Faster than merge sort in practice?
  - Often believed so
  - Does fewer copies and more comparisons, so it depends on the relative cost of these two operations!

Quicksort Overview

1. Pick a pivot element
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot
3. Recursively sort A and C
4. The answer is, "as simple as A, B, C"

Think in Terms of Sets



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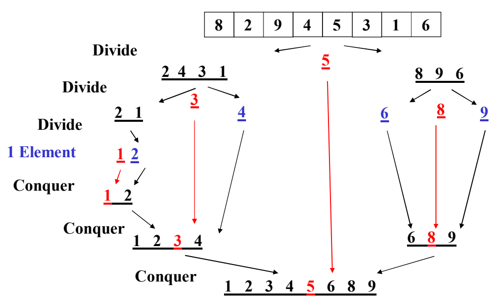
Example, Showing Recursion



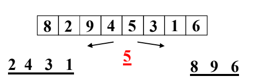
Details

Have not yet explained:

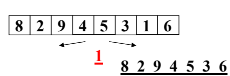
- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about equal in size
- How to implement partitioning
  - In linear time
  - In place

Pivots

- Best pivot?
  - Median
  - Halve each time



- Worst pivot?
  - Greatest/least element
  - Problem of size n - 1
  - $O(n^2)$



Potential pivot rules

While sorting `arr` from `lo` to `hi-1` …

- Pick `arr[lo]` or `arr[hi-1]`
  - Fast, but worst-case occurs with mostly sorted input
- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - Still probably the most elegant approach
- Median of 3, e.g., `arr[lo], arr[hi-1], arr[(hi+lo)/2]`
  - Common heuristic that tends to work well

Partitioning

- Conceptually simple, but hardest part to code up correctly
  - After picking pivot, need to partition in linear time in place
- One approach (there are slightly fancier ones):
  1. Swap pivot with `arr[lo]`
  2. Use two fingers `i` and `j`, starting at `lo+1` and `hi-1`
  3.
```
while (i < j)
    if (arr[j] > pivot) j--
    else if (arr[i] < pivot) i++
    else swap arr[i] with arr[j]
```
  4. Swap pivot with `arr[i]` *

*skip step 4 if pivot ends up being least element

Example

- Step one: pick pivot as median of 3
  - `lo` = 0, `hi` = 10

0	1	2	3	4	5	6	7	8	9
8	1	4	9	0	3	5	2	7	6

- Step two: move pivot to the `lo` position

0	1	2	3	4	5	6	7	8	9
6	1	4	9	0	3	5	2	7	8

Example

Often have more than one swap during partition – this is a short example

Now partition in place

6	1	4	9	0	3	5	2	7	8

Move fingers

6	1	4	9	0	3	5	2	7	8

Swap

6	1	4	2	0	3	5	9	7	8

Move fingers

6	1	4	2	0	3	5	9	7	8

Move pivot

5	1	4	2	0	3	6	9	7	8

Quick sort visualization

- http://www.cs.usfca.edu/~galles/visualization/ComparisonSort.html

Analysis

- Best-case: Pivot is always the median
  - T(0)=T(1)=1
  - T(n)=2T(n/2) + n -- linear-time partition
  - Same recurrence as merge sort: $O(n \log n)$
- Worst-case: Pivot is always smallest or largest element
  - T(0)=T(1)=1
  - T(n) = 1T(n-1) + n
  - Basically same recurrence as selection sort: $O(n^2)$
- Average-case (e.g., with random pivot)
  - $O(n \log n)$, not responsible for proof (in text)

Cutoffs

- For small n, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for large n
- Common engineering technique: switch algorithm below a cutoff
  - Reasonable rule of thumb: use insertion sort for $n < 10$
- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - Switch to sequential algorithm
  - None of this affects asymptotic complexity

Cutoff pseudocode

```
void quicksort(int[] arr, int lo, int hi) {
  if(hi – lo < CUTOFF)
     insertionSort(arr,lo,hi);
  else
     …
}
```

Notice how this cuts out the vast majority of the recursive calls
- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree

How Fast Can We Sort?

- Heapsort & mergesort have $O(n \log n)$ worst-case running time
- Quicksort has $O(n \log n)$ average-case running time
- These bounds are all tight, actually $\Theta(n \log n)$
- Comparison sorting in general is $\Omega(n \log n)$
  - An amazing computer-science result: proves all the clever programming in the world cannot comparison-sort in linear time

The Big Picture

Surprising amount of juicy computer science: 2-3 lectures…

Simple algorithms: $O(n^2)$
- Insertion sort
- Selection sort
- Shell sort
- …

Fancier algorithms: $O(n \log n)$
- Heap sort
- Merge sort
- Quick sort (avg)
- …

Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: $O(n)$
- Bucket sort
- Radix sort

Handling huge data sets
- External sorting

How???
- Change the model – assume more than "compare(a,b)"

Bucket Sort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and *K* (or any small range):
  - Create an array of size *K*
  - Put each element in its proper bucket (a.k.a. bin)
  - *If* data is only integers, no need to store more than a *count* of how times that bucket has been used
- Output result via linear pass through array of buckets

`count` array	
1	3
2	1
3	2
4	2
5	3

- Example:
  K=5
  input (5,1,3,4,3,2,1,1,5,4,5)

  output: 1,1,1,2,3,3,4,4,5,5,5

Visualization

- http://www.cs.usfca.edu/~galles/visualization/CountingSort.html

Analyzing Bucket Sort

- Overall: $O(n+K)$
  - Linear in *n*, but also linear in *K*
  - $\Omega(n \log n)$ lower bound does not apply because this is not a comparison sort
- Good when *K* is smaller (or not much larger) than *n*
  - We don't spend time doing comparisons of duplicates
- Bad when *K* is much larger than *n*
  - Wasted space; wasted time during linear $O(K)$ pass
- For data in addition to integer keys, use list at each bucket

Bucket Sort with Data

- Most real lists aren't just keys; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, insert in *O*(1) (at beginning, or keep pointer to last element)

count array		
1	→	**Rocky V**
2		
3	→	**Harry Potter**
4		
5	→	**Casablanca** → **Star Wars**

- Example: Movie ratings; scale 1-5;1=bad, 5=excellent

  Input=
  - 5: Casablanca
  - 3: Harry Potter movies
  - 5: Star Wars Original Trilogy
  - 1: Rocky V

•Result: 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars
•Easy to keep 'stable'; Casablanca still before Star Wars

Radix sort

- Radix = "the base of a number system"
  - Examples will use 10 because we are used to that
  - In implementations use larger numbers
    - For example, for ASCII strings, might use 128
- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with *least* significant digit
    - Keeping sort *stable*
  - Do one pass per digit
  - Invariant: After *k* passes (digits), the last *k* digits are sorted
- Aside: Origins go back to the 1890 U.S. census

Example

Radix = 10

0	1	2	3	4	5	6	7	8	9
	721		3 143				537 67	478 38	9

Input: 478, 537, 9, 721, 3, 38, 143, 67

First pass:
bucket sort by ones digit

Order now: 721, 3, 143, 537, 67, 478, 38, 9

Example

0	1	2	3	4	5	6	7	8	9
	721		3 143				537 67	478 38	9

Radix = 10

0	1	2	3	4	5	6	7	8	9
3 9		721	537 38	143		67	478		

Order was: 721, 3, 143, 537, 67, 478, 38, 9

Second pass:
stable bucket sort by tens digit

Order now: 3, 9, 721, 537, 38, 143, 67, 478

Example

0	1	2	3	4	5	6	7	8	9
3 9		721	537 38	143		67	478		

Radix = 10

0	1	2	3	4	5	6	7	8	9
3 9 38 67	143			478	537		721		

Order was: 3, 9, 721, 537, 38, 143, 67, 478

Third pass:
stable bucket sort by 100s digit

Order now: 3, 9, 38, 67, 143, 478, 537, 721

Visualization

- http://www.cs.usfca.edu/~galles/visualization/RadixSort.html

Analysis

Input size: n
Number of buckets = Radix: B
Number of passes = "Digits": P

Work per pass is 1 bucket sort: $O(B+n)$

Total work is $O(P(B+n))$

Compared to comparison sorts, sometimes a win, but often not
- Example: Strings of English letters up to length 15
  - Run-time proportional to: $15*(52 + n)$
  - This is less than n log n only if $n > 33{,}000$
  - Of course, cross-over point depends on constant factors of the implementations
    - And radix sort can have poor locality properties

Sorting massive data

- Need sorting algorithms that minimize disk/tape access time:
  - Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
  - Merge sort scans linearly through arrays, leading to (relatively) efficient sequential disk access
- Merge sort is the basis of massive sorting
- Merge sort can leverage multiple disks

External Merge Sort

- Sort 900 MB using 100 MB RAM
  - Read 100 MB of data into memory
  - Sort using conventional method (e.g. quicksort)
  - Write sorted 100MB to temp file
  - Repeat until all data in sorted chunks (900/100 = 9 total)
- Read first 10 MB of each sorted chuck, merge into remaining 10MB
  - writing and reading as necessary
  - Single merge pass instead of *log n*
  - Additional pass helpful if data much larger than memory
- Parallelism and better hardware can improve performance
- Distribution sorts (similar to bucket sort) are also used

Last Slide on Sorting

- Simple $O(n^2)$ sorts can be fastest for small n
  - Selection sort, Insertion sort (latter linear for mostly-sorted)
  - Good for "below a cut-off" to help divide-and-conquer sorts
- $O(n \log n)$ sorts
  - Heap sort, in-place but not stable nor parallelizable
  - Merge sort, not in place but stable and works as external sort
  - Quick sort, in place but not stable and $O(n^2)$ in worst-case
    - Often fastest, but depends on costs of comparisons/copies
- $\Omega(n \log n)$ is worst-case and average lower-bound for sorting by comparisons
- Non-comparison sorts
  - Bucket sort good for small number of possible key values
  - Radix sort uses fewer buckets and more phases
- Best way to sort? It depends!