CSE373: Data Structures and Algorithms

Implicit Graphs

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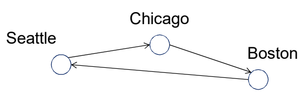
Explicit vs Implicit Graphs

- Explicit Graph:
  - All vertices are identified individually and represented separately.
  - All edges are identified individually and represented separately.
- Implicit Graph:
  - Only a subset, possibly only one, of the vertices is given an explicit representation. (The others are implied.)
  - Only a subset, and possibly zero, of the edges is given an explicit representation.
  - A set of “operators” is provided that can be used to construct “new” edges and vertices.

Example Explicit Graph

G = (V, E), V = {Seattle, Chicago, Boston},
E = {(Seattle, Chicago), (Chicago, Boston), (Boston, Seattle)}



Example Implicit Graph

G = (V, E), V = {v_0, v_1, …, v_9},
v_0 = {0}

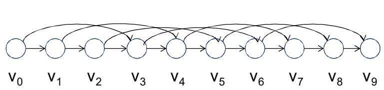
Operators: { φ_0, φ_1}

φ_0 precondition: i < 9
transition: $f_0(v_i) = f_0(\{i\}) = \{i+1\}$
$v_j = \varphi_0(\{i\})$ is a (possibly) new vertex
(v_i , v_j) is a (possibly) new edge

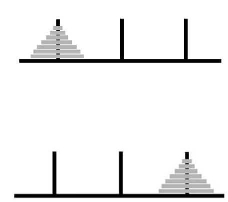
φ_1 precondition: i < 7
transition: $f_1(v_i) = f_1(\{i\}) = \{i+3\}$
$v_j = \varphi_1(\{i\})$ is a (possibly) new vertex
(v_i , v_j) is a (possibly) new edge

Some Implicit Graphs Can Be Made Explicit

- G = (V, E)
- V = {{0}, …, {9}}
- E = {({0} , {1}), ({1} , {2}), ({2} , {3}), … , ({8} , {9}),
  ({0} , {3}), ({1} , {4}), ({2} , {5}), … , ({6} , {9}) } 10 + 7 = 17 edges



The Towers of Hanoi (Puzzle)



Example: TOH Graph*

$G = (V, E)$
$v_0 = [[4321],[\ ], [\ ]]$

$v_k = [p_0, p_1, p_2]$; $p_m = [d_{m,0}, d_{m,1}, \ldots, d_{m,n_m}]$

$\varphi_{i,j}$ precondition : p_i != [] and
if p_j != [] then $d_{i,n_i} < d_{j,n_j}$

$\varphi_{i,j}$ transition :
p_i: $[\alpha, d_{i,n_i}]$, p_j: $[\beta] \Rightarrow p_i$: $[\alpha]$, p_j: $[\beta, d_{i,n_i}]$

The precondition is that peg i must have at least one disk, and if peg j has any disks, the top (last) disk on peg i must be smaller than the last disk on peg j.

The transition is that the top disk on peg i is removed from peg i and put on the top of the pile on peg j.

* Towers of Hanoi

TOH Graph – properties

G_n = TOH graph for an n-disk TOH puzzle.

$G_n = (V_n, E_n)$

What is $|V_n|$?

What is $|E_n|$?

Diameter of G_n ?

Longest distance between two vertices of the graph.

TOH Graph – properties

G_n = TOH graph for an n-disk TOH puzzle. **n=1**

$G_n = (V_n, E_n)$

What is $|V_n|$? **3**

What is $|E_n|$? **3**

Diameter of G_n ? **1**

Longest distance between two vertices of the graph.

TOH Graph – properties

G_n = TOH graph for an n-disk TOH puzzle. n=1, n=2

$G_n = (V_n, E_n)$

What is $|V_n|$? 3 9

What is $|E_n|$? 3 12

Diameter of G_n ? 1 3

Longest distance between two vertices of the graph.

TOH Graph – properties

G_n = TOH graph for an n-disk TOH puzzle. n=1, n=2, **n=3**

$G_n = (V_n, E_n)$

What is $|V_n|$? 3 9 **27**

What is $|E_n|$? 3 12 **39**

Diameter of G_n ? 1 3 **7**

Longest distance between two vertices of the graph.

TOH Graph – properties

G_n = TOH graph for an n-disk TOH puzzle. n=1, n=2, n=3, **n=4**

$G_n = (V_n, E_n)$

What is $|V_n|$? 3 9 27 **81**

What is $|E_n|$? 3 12 39 **120**

Diameter of G_n ? 1 3 7 15

Longest distance between two vertices of the graph.

TOH Graph – properties

G_n = TOH graph for an n-disk TOH puzzle.	n=1, n=2, n=3, n=4
$G_n = (V_n, E_n)$	
What is $\vert V_n \vert$? $\vert \mathbf{V_n} \vert = 3 \cdot \vert \mathbf{V_{n-1}} \vert$	3 9 27 81
What is $\vert E_n \vert$?	3 12 39 120
Diameter of G_n ?	1 3 7 15
Longest distance between two vertices of the graph.	

TOH Graph – properties

G_n = TOH graph for an n-disk TOH puzzle.	n=1, n=2, n=3, n=4
$G_n = (V_n, E_n)$	
What is $\vert V_n \vert$? $\vert \mathbf{V_n} \vert = 3 \cdot \vert \mathbf{V_{n-1}} \vert$	3 9 27 81
What is $\vert E_n \vert$? $\vert \mathbf{E_n} \vert = 3 \cdot \vert \mathbf{V_n} \vert + 3$	3 12 39 120
Diameter of G_n ?	1 3 7 15
Longest distance between two vertices of the graph.	

TOH Graph – properties

G_n = TOH graph for an n-disk TOH puzzle.	n=1, n=2, n=3, n=4
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Diameter of G_n ? $\mathbf{D(G_n) = 2 \cdot D(G_{n-1}) + 1}$	1 3 7 15
Longest distance between two vertices of the graph.	

TOH Graph – properties

G_n = TOH graph for an n-disk TOH puzzle.	n=1, n=2, n=3, n=4
$G_n = (V_n, E_n)$	
What is $\vert V_n \vert$? $\vert \mathbf{V_n} \vert = 3 \cdot \vert \mathbf{V_{n-1}} \vert$	3 9 27 81
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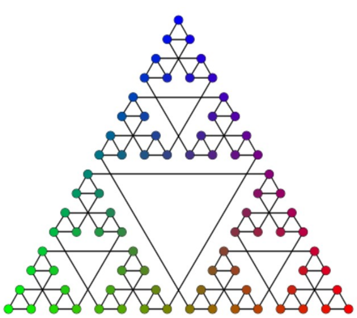
Longest distance between two vertices of the graph.	

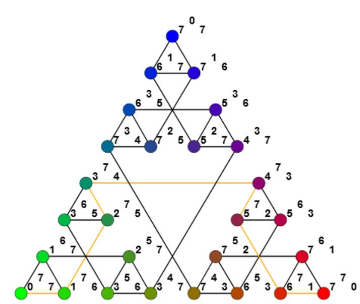
n_disks = 4



n_disks = 3

With heuristic distances to landmarks.

Solution path shown (the golden path)



Creating Graph Layouts

- Each vertex should be assigned a location on the “page” that helps reflect the structure of the graph.
- One method is based on “landmark vertices.”
  - Formulate a distance function $d(v_0, v_1)$.
  - Identify 3 or more “landmark” states in Sigma: $\{L_1, L_2, L_3, \ldots, L_n\}$ and assign them (x,y) locations.
  - For vertex v, use d to find barycentric coordinates $(b_1, b_2, \ldots, b_n)$ in terms of $L_1, \ldots, L_n$, and plot v.

Heuristic distance $d_h(v_0, v_1)$

Natural method: weight larger disks more heavily. Then add up the absolute values of differences.

$$d_h(v_0, v_1) = \sum_{k=0}^{n-1} 2^k \,|\, q_k(v_0, v_1) \,|$$

Where $q_k(v_0, v_1) = 1$ if disk k is on different pegs in v_0 and v_1, and 0 otherwise.

Disk 0 is the smallest disk. (n is the number of disks.)

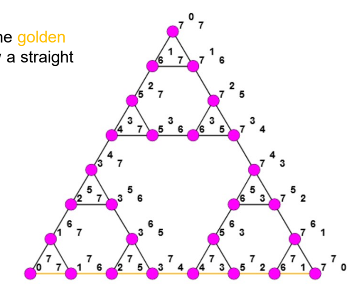
Layout depends on distance function

Let $d_m(v_0, v_1)$ = minimum number of moves needed to reach v_0 from v_1.

Then the solution path can be a straight line.

TOH Graph Laid Out with d_m and barycentric coordinates.

Note that the golden path is now a straight line.



Summary

Graphs can be represented explicitly, implicitly, or with a combination of methods.

Some graphs come in families that share properties.
e.g., TOH graph $G_4 \in \{G_n$ such that $n > 0\}$

Graph layout can reveal structural information.

Problems and puzzles can be represented by graphs.

Path-finding methods can be used to solve problems.