CSE373: Data Structures and Algorithms

Priority Queues and Binary Heaps

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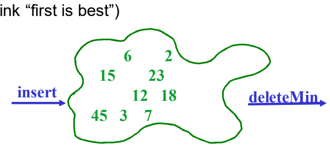
This lecture material represents the work of multiple instructors at the University of Washington. Thank you to all who have contributed!

A new ADT: Priority Queue

- A **priority queue** holds *compare-able data (totally ordered)*
  - Like dictionaries with ordered keys, we need to *compare items*
    - Given *x* and *y*, is *x* less than, equal to, or greater than *y*
    - Meaning of the ordering can depend on your data
  - Integers are comparable, so we'll use them in examples
    - But the priority queue ADT is much more general
    - Typically two fields, the *priority* and the *data*

Priorities

- Each item has a "priority"
  - In our examples, the *lesser* item is the one with the *greater* priority
  - So "priority 1" is more important than "priority 4"
  - (Just a convention, think "first is best")
- Operations:
  - `insert`
  - `deleteMin`
  - `is_empty`



- Key property: `deleteMin` *returns* and *deletes* the item with greatest priority (lowest priority value)
  - Can resolve ties arbitrarily

Example

`insert` *x1* with priority *5*
`insert` *x2* with priority *3*
`insert` *x3* with priority *4*
a = `deleteMin // x2`
b = `deleteMin // x3`
`insert` *x4* with priority *2*
`insert` *x5* with priority *6*
c = `deleteMin // x4`
d = `deleteMin // x1`

- Analogy: `insert` is like `enqueue`, `deleteMin` is like `dequeue`
  - But the whole point is to use priorities instead of FIFO

Applications

Like all good ADTs, the priority queue arises often
- Sometimes blatant, sometimes less obvious

- Run multiple programs in the operating system
  - "critical" before "interactive" before "compute-intensive"
  - Maybe let users set priority level
- Treat hospital patients in order of severity (or triage)
- Select print jobs in order of decreasing length?
- Forward network packets in order of urgency
- Select most frequent symbols for data compression
- Sort (first `insert` all, then repeatedly `deleteMin`)

Finding a good data structure

- Will show an efficient, non-obvious data structure for this ADT
  - But first let's analyze some "obvious" ideas for *n* data items
  - All times worst-case; assume arrays "have room"

data	*insert algorithm / time*		*deleteMin algorithm / time*	
unsorted array	add at end	$O(1)$	search	$O(n)$
unsorted linked list	add at front	$O(1)$	search	$O(n)$
sorted circular array	search / shift	$O(n)$	move front	$O(1)$
sorted linked list	put in right place	$O(n)$	remove at front	$O(1)$
binary search tree	put in right place	$O(n)$	leftmost	$O(n)$
AVL tree	put in right place	$O(\log n)$	leftmost	$O(\log n)$

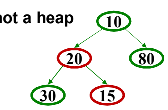
More on possibilities

- One more idea: if priorities are 0, 1, …, *k* can use an array of k lists
  - `insert`: add to front of list at `arr[priority]`, *O*(1)
  - `deleteMin`: remove from lowest non-empty list *O(k)*
- We are about to see a data structure called a "binary heap"
  - Another binary tree structure with specific properties
  - *O*(`log` *n*) `insert` and *O*(`log` *n*) `deleteMin` *worst-case*
    - Possible because we don't support unneeded operations; no need to maintain a full sort
  - *Very* good constant factors
  - *If* items arrive in random order, then `insert` is *O*(1) on *average*
    - Because 75% of nodes in bottom two rows

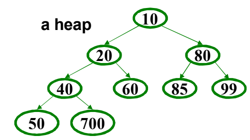
Our data structure

A *binary min-heap* (or just *binary heap* or just *heap*) has:
- Structure property: A *complete* binary tree
- Heap property: The priority of every (non-root) node is less important than the priority of its parent
  - ***Not** a binary search tree*

not a heap



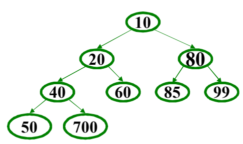
a heap



So:
- Where is the highest-priority item?
- What is the height of a heap with *n* items?

Operations: basic idea

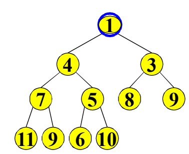
- `findMin`: return `root.data`
- `deleteMin`:
  1. `answer = root.data`
  2. Move right-most node in last row to root to restore structure property
  3. "Percolate down" to restore heap property
- `insert:`
  1. Put new node in next position on bottom row to restore structure property
  2. "Percolate up" to restore heap property



Overall strategy:
- *Preserve structure property*
- *Break and restore heap property*

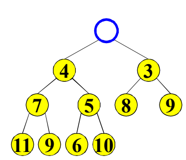
DeleteMin

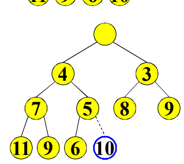
Delete (and later return) value at root node



DeleteMin: Keep the Structure Property

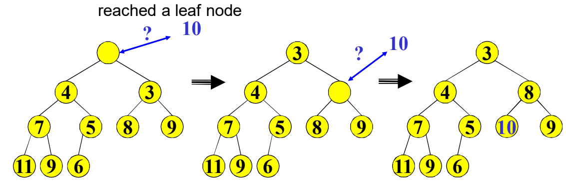
- We now have a "hole" at the root
  - Need to fill the hole with another value
- Keep structure property: When we are done, the tree will have one less node and must still be complete
- Pick the last node on the bottom row of the tree and move it to the "hole"





DeleteMin: Restore the Heap Property

Percolate down:
- Keep comparing priority of item with both children
- If priority is less important, swap with the most important child and go down one level
- Done if both children are less important than the item or we've reached a leaf node



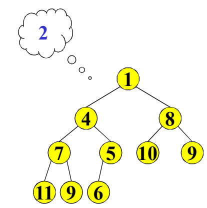
Why is this correct?
What is the run time?

DeleteMin: Run Time Analysis

- Run time is *O*(height of heap)
- A heap is a complete binary tree
- Height of a complete binary tree of *n* nodes?
  - height = $\lfloor \log_2(n) \rfloor$
- Run time of `deleteMin` is $O(\log n)$

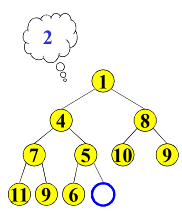
Insert

- Add a value to the tree
- Afterwards, structure and heap properties must still be correct



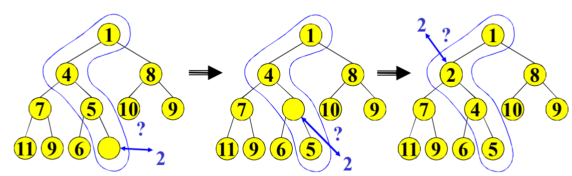
Insert: Maintain the Structure Property

- There is only one valid tree shape after we add one more node
- So put our new data there and then focus on restoring the heap property



Insert: Restore the heap property

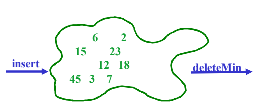
Percolate up:
- Put new data in new location
- If parent is less important, swap with parent, and continue
- Done if parent is more important than item or reached root



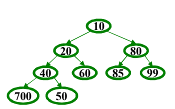
What is the running time?
Like `deleteMin`, worst-case time proportional to tree height: *O*(log *n*)

Summary

- Priority Queue ADT:
  - `insert` comparable object,
  - `deleteMin`



- Binary heap data structure:
  - Complete binary tree
  - Each node has less important priority value than its parent



- `insert` and `deleteMin` operations = *O*(height-of-tree)=*O*(`log` *n*)
  - `insert`: put at new last position in tree and percolate-up
  - `deleteMin`: remove root, put last element at root and percolate-down

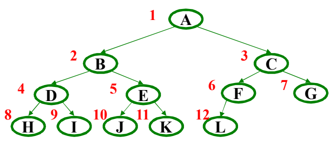
Efficiently Implementing the Priority Queue ADT

By using a special data structure called a *binary heap**, we will achieve:

- In-place data representation (no links, no wasted fields).
- Fast `insert` and `deleteMin` operations -- *O(log n)*
- Fast initialization of an n-element priority queue (the `buildHeap` operation) – *O(n)*

*Be careful not to confuse the binary heap data structure with other meanings of "heap" as in "runtime heap" (a pool of memory locations available for dynamic allocation).

Array Representation of Binary Trees



From node `i`:

left child: `i*2`
right child: `i*2+1`
parent: `i/2`

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Judging the array implementation

Plusses:
- Non-data space: just index 0 and unused space on right
  - In conventional tree representation, one edge per node (except for root), so *n*-1 wasted space (like linked lists)
  - Array would waste more space if tree were not complete
- Multiplying and dividing by 2 is very fast (shift operations in hardware)
- Last used position is just index `size`

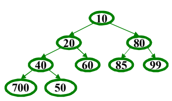
Minuses:
- Same might-be-empty or might-get-full problems we saw with stacks and queues (resize by doubling as necessary)

Plusses outweigh minuses: “this is how people do it”

Pseudocode: insert into binary heap

```
def insert(val):
  if size==len(arr)-1:
    resize()
  size += 1
  i=percolateUp(size,val)
  arr[i] = val
```

```
def percolateUp(hole, val):
  while(hole > 1 &&
        val < arr[hole/2]):
    arr[hole] = arr[hole/2]
    hole /= 2
  return hole
```

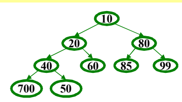


	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Pseudocode: deleteMin from binary heap

```
def deleteMin():
  if isEmpty() raise…
  ans = arr[1]
  hole = percolateDown
          (1,arr[size])
  arr[hole] = arr[size]
  size -= 1
  return ans
```

```
def percolateDown(hole,val):
 while(2*hole <= size):
  left  = 2*hole
  right = left + 1
  if right > size ||
     arr[left] < arr[right]:
    target = left
  else:
    target = right
  if arr[target] < val:
    arr[hole] = arr[target]
    hole = target
  else: break
return hole
```

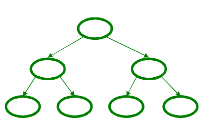


	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

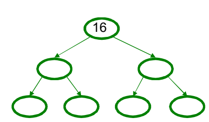
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

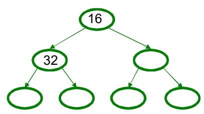
	16						
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

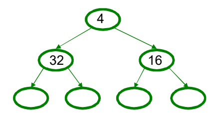
	16	**32**					
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

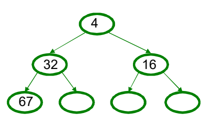
	4	**32**	**16**				
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

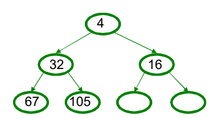
	4	**32**	**16**	**67**			
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

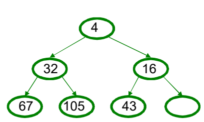
	4	**32**	**16**	**67**	**105**		
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

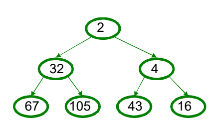
	4	**32**	**16**	**67**	**105**	**43**	
0	1	2	3	4	5	6	7



Example

1. insert: 16, 32, 4, 67, 105, 43, 2
2. deleteMin

	2	**32**	**4**	**67**	**105**	**43**	**16**
0	1	2	3	4	5	6	7



Other operations

- **`decreaseKey`**: given pointer to object in priority queue (e.g., its array index), lower its priority value by *p*
  - Change priority and percolate up
- **`increaseKey`**: given pointer to object in priority queue (e.g., its array index), raise its priority value by *p*
  - Change priority and percolate down
- **`remove`**: given pointer to object in priority queue (e.g., its array index), remove it from the queue
  - **`decreaseKey`** with $p = \infty$, then **`deleteMin`**

Running time for all these operations?

Build Heap

- Suppose you have *n* items to put in a new (empty) priority queue
  - Call this operation **`buildHeap`**
- *n* **`insert`**s works
  - Only choice if ADT doesn't provide **`buildHeap`** explicitly
  - *O*(*n* **`log`** *n*)
- Why would an ADT provide this unnecessary operation?
  - Convenience
  - Efficiency: an *O*(*n*) algorithm called Floyd's Method
  - Common issue in ADT design: how many specialized operations

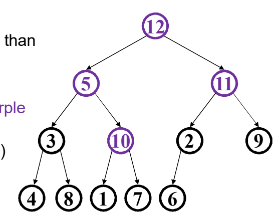
Floyd's Method

1. Use *n* items to make any complete tree you want
   - That is, put them in array indices 1,…,*n*
2. Treat it as a heap and fix the heap-order property
   - Bottom-up: leaves are already in heap order, work up toward the root one level at a time

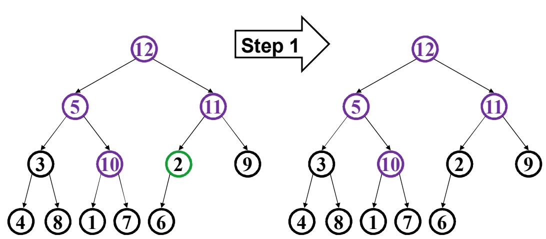
```
def buildHeap():
  for i in range(size/2, -1, -1):
    val  = arr[i]
    hole = percolateDown(i,val)
    arr[hole] = val
```


Example

- In tree form for readability
  - Purple for node not less than descendants
    - heap-order problem
  - Notice no leaves are purple
  - Check/fix each non-leaf bottom-up (6 steps here)

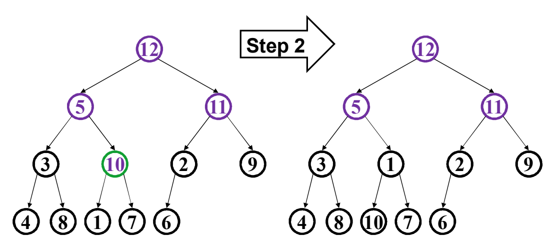


Example



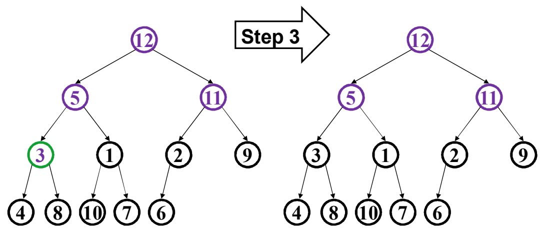
- Happens to already be less than children (er, child)

Example



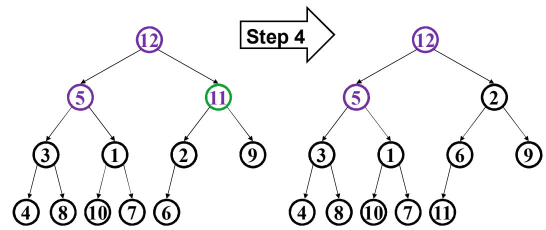
- Percolate down (notice that moves 1 up)

Example



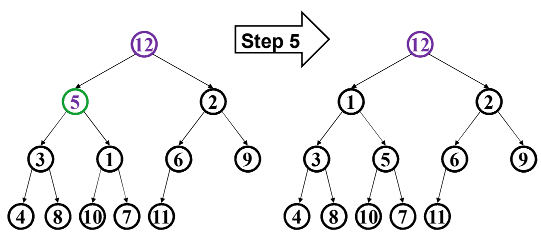
- Another nothing-to-do step

Example

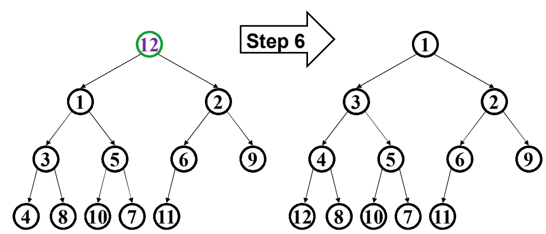


- Percolate down as necessary (steps 4a and 4b)

Example



Example



But is it right?

- “Seems to work”
  - Let’s *prove* it restores the heap property (correctness)
  - Then let’s *prove* its running time (efficiency)

```
def buildHeap():
  for i in range(size/2, -1, -1):
    val  = arr[i]
    hole = percolateDown(i,val)
    arr[hole] = val
```


Correctness

```
def buildHeap():
  for i in range(size/2, -1, -1):
    val  = arr[i]
    hole = percolateDown(i,val)
    arr[hole] = val
```

Loop Invariant: For all `j>i`, `arr[j]` is less than its children

- True initially: If `j > size/2`, then `j` is a leaf
  - Otherwise its left child would be at position > `size`
- True after one more iteration: loop body and `percolateDown` make `arr[i]` less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

Efficiency

```
def buildHeap():
  for i in range(size/2, -1, -1):
    val  = arr[i]
    hole = percolateDown(i,val)
    arr[hole] = val
```

Easy argument: **buildHeap** is $O(n \log n)$ where n is **size**

- **size/2** loop iterations
- Each iteration does one **percolateDown**, each is $O(\log n)$

This is correct, but there is a more precise ("tighter") analysis of the algorithm…

Efficiency

```
def buildHeap():
  for i in range(size/2, -1, -1):
    val  = arr[i]
    hole = percolateDown(i,val)
    arr[hole] = val
```

Better argument: **buildHeap** is $O(n)$ where n is **size**

- **size/2** total loop iterations: $O(n)$
- 1/2 the loop iterations percolate at most 1 step
- 1/4 the loop iterations percolate at most 2 steps
- 1/8 the loop iterations percolate at most 3 steps
- …
- ((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + …) < 2 (page 4 of Weiss)
  - So at most **2(size/2)** *total* percolate steps: $O(n)$

Lessons from **buildHeap**

- Without **buildHeap**, our ADT already let clients implement their own in $O(n \log n)$ worst case
- By providing a specialized operation internal to the data structure (with access to the internal data), we can do $O(n)$ worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness:
    - Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was $O(n \log n)$
    - Tighter analysis shows same algorithm is $O(n)$

Other branching factors

- *d*-heaps: have *d* children instead of 2
  - Makes heaps shallower, useful for heaps too big for memory (or cache)
- Example: a 3-heap
  - Just have three children instead of 2
  - Still use an array with all positions from 1…heap-size used

Index	Children Indices
1	2,3,4
2	5,6,7
3	8,9,10
4	11,12,13
5	14,15,16
…	…