CSE373: Data Structures and Algorithms

# Comparison Sorting

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This lecture material represents the work of multiple instructors at the University of Washington. Thank you to all who have contributed!

## Introduction to Sorting

- Stacks, queues, priority queues, and dictionaries all focused on providing one element at a time
- But often we know we want "all the things" in some order
  - Humans can sort, but computers can sort fast
  - Very common to need data sorted somehow
    - Alphabetical list of people
    - List of countries ordered by population
    - Search engine results by relevance
    - …
- Algorithms have different asymptotic and constant-factor trade-offs
  - No single "best" sort for all scenarios
  - Knowing one way to sort just isn't enough



## More Reasons to Sort

General technique in computing:
*Preprocess data to make subsequent operations faster*

Example: Sort the data so that you can
- Find the $k^{th}$ largest in constant time for any **k**
- Perform binary search to find elements in logarithmic time

Whether the performance of the preprocessing matters depends on
- How often the data will change (and how much it will change)
- How much data there is



## Why Study Sorting in this Class?

- You might never need to reimplement a sorting algorithm yourself
  - Standard libraries will generally implement one or more (Java implements 2)
- You will almost certainly use sorting algorithms
  - Important to understand relative merits and expected performance
- Excellent set of algorithms for practicing analysis and comparing design techniques
  - Classic part of a data structures class, so you'll be expected to know it



## The main problem, stated carefully

For now, assume we have *n* comparable elements in an array and we want to rearrange them to be in increasing order

Input:
- An array **A** of data records
- A key value in each data record
- A comparison function (consistent and total)

Effect:
- Reorganize the elements of **A** such that for any **i** and **j**, if **i < j** then **A[i] ≤ A[j]**
- (Also, **A** must have exactly the same data it started with)
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort



## Variations on the Basic Problem

1. Maybe elements are in a linked list (could convert to array and back in linear time, but some algorithms needn't do so)
2. Maybe ties need to be resolved by "original array position"
   - Sorts that do this naturally are called stable sorts
   - Others could tag each item with its original position and adjust comparisons accordingly (non-trivial constant factors)
3. Maybe we must not use more than $O(1)$ "auxiliary space"
   - Sorts meeting this requirement are called in-place sorts
4. Maybe we can do more with elements than just compare
   - Sometimes leads to faster algorithms
5. Maybe we have too much data to fit in memory
   - Use an "external sorting" algorithm

## Sorting: The Big Picture

Surprising amount of neat stuff to say about sorting:

| Simple algorithms: $O(n^2)$ | Fancier algorithms: $O(n \log n)$ | Comparison lower bound: $\Omega(n \log n)$ | Specialized algorithms: $O(n)$ | Handling huge data sets |
|---|---|---|---|---|
| Insertion sort<br>Selection sort<br>Shell sort<br>… | Heap sort<br>Merge sort<br>Quick sort<br>… | | Bucket sort<br>Radix sort | External sorting |



## Insertion Sort

- Idea: At step `k`, put the `k`th element in the correct position among the first `k` elements
- Alternate way of saying this:
  - Sort first two elements
  - Now insert 3rd element in order
  - Now insert 4th element in order
  - …
- "Loop invariant": when loop index is `i`, first `i` elements are sorted
- Let's see a visualization (http://www.cs.usfca.edu/~galles/visualization/ComparisonSort.html)
- Time?
  Best-case _____ Worst-case _____ "Average" case ____



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- Time?
  Best-case $O(n)$ Worst-case $O(n^2)$ "Average" case $O(n^2)$
  start sorted start reverse sorted (see text)



## Selection sort

- Idea: At step `k`, find the smallest element among the not-yet-sorted elements and put it at position k
- Alternate way of saying this:
  - Find smallest element, put it 1st
  - Find next smallest element, put it 2nd
  - Find next smallest element, put it 3rd …
- "Loop invariant": when loop index is `i`, first `i` elements are the `i` smallest elements in sorted order
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- Time?
  Best-case $O(n^2)$ Worst-case $O(n^2)$ "Average" case $O(n^2)$
  *Always* $T(1) = 1$ and $T(n) = n + T(n-1)$



## Insertion Sort vs. Selection Sort

- Different algorithms
- Solve the same problem
- Have the same worst-case and average-case asymptotic complexity
  - Insertion-sort has better best-case complexity; preferable when input is "mostly sorted"
- Other algorithms are more efficient *for large arrays that are not already almost sorted*
  - Insertion sort may do well on small arrays

## *Aside: We Will Not Cover Bubble Sort*

- It is not, in my opinion, what a "normal person" would think of
- It doesn't have good asymptotic complexity: $O(n^2)$
- It's not particularly efficient with respect to constant factors

Basically, almost everything it is good at some other algorithm is at least as good at

- Perhaps people teach it just because someone taught it to them?

Fun, short, optional read:
*Bubble Sort: An Archaeological Algorithmic Analysis*, Owen Astrachan, SIGCSE 2003, http://www.cs.duke.edu/~ola/bubble/bubble.pdf



## *The Big Picture*

Surprising amount of juicy computer science: 2-3 lectures…

| **Simple algorithms: $O(n^2)$** | **Fancier algorithms: $O(n \log n)$** | **Comparison lower bound: $\Omega(n \log n)$** | **Specialized algorithms: $O(n)$** | **Handling huge data sets** |
|---|---|---|---|---|
| Insertion sort<br>Selection sort<br>Shell sort<br>… | Heap sort<br>Merge sort<br>Quick sort (avg)<br>… | | Bucket sort<br>Radix sort | External sorting |



## *Heap sort*

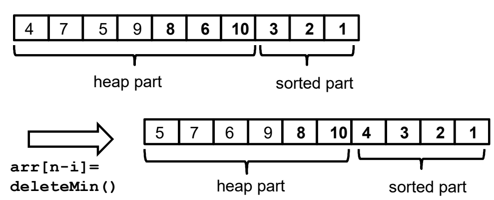
- Sorting with a heap is easy:
  - `insert` each `arr[i]`, or better yet use `buildHeap`
  - ```
    for i in range(len(arr)):
        arr[i] = deleteMin()
    ```
- Worst-case running time: $O(n$ `log` $n)$
- We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There's a trick to make it in-place…



## *In-place heap sort*

But this reverse sorts – how would you fix that?

- Treat the initial array as a heap (via `buildHeap`)
- When you delete the `i`th element, put it at `arr[n-i]`
  - That array location isn't needed for the heap anymore!






## *"AVL sort"*

- We can also use a balanced tree to:
  - `insert` each element: total time $O(n$ `log` $n)$
  - Repeatedly `deleteMin`: total time $O(n$ `log` $n)$
    - Better: in-order traversal $O(n)$, but still $O(n$ `log` $n)$ overall
- But this cannot be made in-place and has worse constant factors than heap sort
  - both are $O(n$ `log` $n)$ in worst, best, and average case
  - neither parallelizes well
  - heap sort is better



## *"Hash sort"???*

- Don't even think about trying to sort with a hash table!
- Finding min item in a hashtable is $O(n)$, so this would be a slower, more complicated selection sort
- And we've already seen that selection sort is pretty bad!
    ```

## Divide and conquer

Very important technique in algorithm design

1. Divide problem into smaller parts
2. Independently solve the simpler parts
   – Think recursion
   – Or potential parallelism
3. Combine solution of parts to produce overall solution

(This technique has a *long* history.)



## Divide-and-Conquer Sorting

Two great sorting methods are fundamentally divide-and-conquer

1. Mergesort: Sort the left half of the elements (recursively)
   Sort the right half of the elements (recursively)
   Merge the two sorted halves into a sorted whole
   (we covered Mergesort early in the quarter)
2. Quicksort: Pick a "pivot" element
   Divide elements into less-than pivot
   and greater-than pivot
   Sort the two divisions (recursively on each)
   Answer is:
   **sorted-less-than** then **pivot** then **sorted-greater-than**

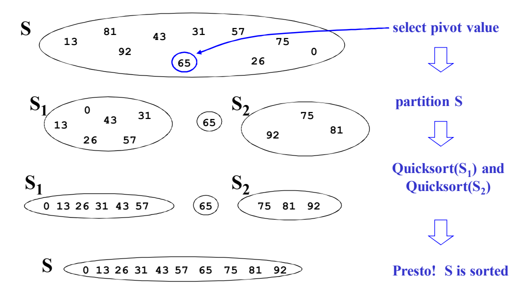


## Quicksort Overview

1. Pick a pivot element
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot
3. Recursively sort A and C
4. The answer is, "as simple as A, B, C"



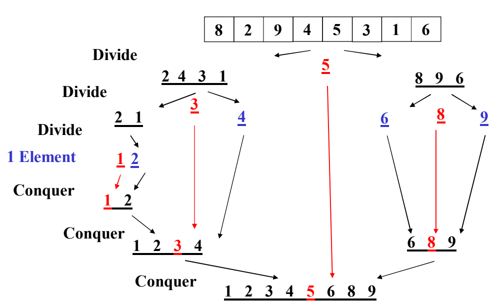
## Think in Terms of Sets




[Weiss]



## Example, Showing Recursion






## Details

Have not yet explained:

- How to pick the pivot element
  – Any choice is correct: data will end up sorted
  – But as analysis will show, want the low and high subsets to be about equal in size
- How to implement partitioning
  – In linear time
  – In place

## Pivots

- Best pivot?
  - Median
  - Halve each time

| 8 | 2 | 9 | 4 | 5 | 3 | 1 | 6 |
|---|---|---|---|---|---|---|---|

2 4 3 1    5    8 9 6

- Worst pivot?
  - Greatest/least element
  - Problem of size n - 1
  - $O(n^2)$

| 8 | 2 | 9 | 4 | 5 | 3 | 1 | 6 |
|---|---|---|---|---|---|---|---|

1    8 2 9 4 5 3 6



## Potential pivot rules

While sorting `arr` from `lo` to `hi-1` …

- Pick `arr[lo]` or `arr[hi-1]`
  - Fast, but worst-case occurs with mostly sorted input
- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - Still probably the most elegant approach
- Median of 3, e.g., `arr[lo], arr[hi-1], arr[(hi+lo)/2]`
  - Common heuristic that tends to work well



## Partitioning

- Conceptually simple, but hardest part to code up correctly
  - After picking pivot, need to partition in linear time in-place
- One approach (there are slightly fancier ones):
  1. Swap pivot with `arr[lo]`
  2. Use two cursors `i` and `j`, starting at `lo+1` and `hi-1`
  3. 
```
while (i < j)
    if (arr[j] > pivot) j--
    else if (arr[i] < pivot) i++
    else swap arr[i] with arr[j]
```
  4. Swap pivot with `arr[i]` *

*skip step 4 if pivot ends up being least element



## Example

- Step one: pick pivot as median of 3
  - `lo` = 0, `hi` = 10

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1 | 4 | 9 | 0 | 3 | 5 | 2 | 7 | 6 |

- Step two: move pivot to the `lo` position

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 1 | 4 | 9 | 0 | 3 | 5 | 2 | 7 | 8 |



## Example

Often have more than one swap during partitioning – this is a short example

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Now partition in place | 6 | 1 | 4 | 9 | 0 | 3 | 5 | 2 | 7 | 8 |
| Move fingers | 6 | 1 | 4 | 9 | 0 | 3 | 5 | 2 | 7 | 8 |
| Swap | 6 | 1 | 4 | 2 | 0 | 3 | 5 | 9 | 7 | 8 |
| Move fingers | 6 | 1 | 4 | 2 | 0 | 3 | 5 | 9 | 7 | 8 |
| Move pivot | 5 | 1 | 4 | 2 | 0 | 3 | 6 | 9 | 7 | 8 |



## Quick sort visualization

- http://www.cs.usfca.edu/~galles/visualization/ComparisonSort.html

## Analysis

- Best-case: Pivot is always the median
  T(0) = T(1)=1
  T($n$) = 2 T($n$/2) + $n$ -- linear-time partitioning
  Same recurrence as merge sort: $O(n \log n)$
- Worst-case: Pivot is always smallest or largest element
  T(0) = T(1)=1
  T($n$) = 1 T($n$-1) + $n$
  Basically same recurrence as selection sort: $O(n^2)$
- Average-case (e.g., with random pivot)
  - O($n$ `log` $n$), not responsible for proof (in text)



## Cutoffs

- For small $n$, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for large $n$
- Common engineering technique: switch algorithm below a cutoff
  - Reasonable rule of thumb: use insertion sort for $n < 10$
- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - Switch to sequential algorithm
  - None of this affects asymptotic complexity



## Cutoff pseudocode

```
void quicksort(int[] arr, int lo, int hi) {
  if(hi – lo < CUTOFF)
     insertionSort(arr,lo,hi);
  else
     …
}
```

Notice how this cuts out the vast majority of the recursive calls
- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree