CSE373: Data Structures and Algorithms

Shortest Paths and Dijkstra's Algorithm

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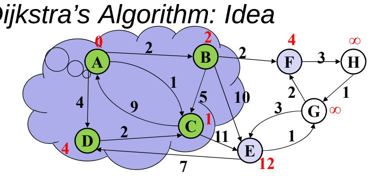
Autumn 2016

This lecture material represents the work of multiple instructors at the University of Washington. Thank you to all who have contributed!

Dijkstra's Algorithm

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a "best distance so far"
  - A priority queue will turn out to be useful for efficiency
- An example of a greedy algorithm
  - A series of steps
  - At each one the locally optimal choice is made

Dijkstra's Algorithm: Idea



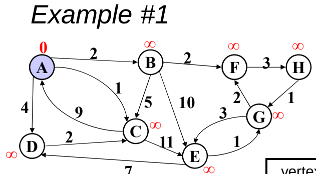
- Initially, start node has cost 0 and all other nodes have cost ∞
- At each step:
  - Pick closest unknown vertex **v**
  - Add it to the "cloud" of known vertices
  - Update distances for nodes with edges from **v**
- That's it! (But we need to prove it produces correct answers)

The Algorithm

1. For each node **v**, set `v.cost = ∞` and `v.known = false`
2. Set `source.cost = 0`
3. While there are unknown nodes in the graph
   a) Select the unknown node **v** with lowest cost
   b) Mark **v** as known
   c) For each edge `(v,u)` with weight `w`,

```
c1 = v.cost + w    // cost of best path through v to u
c2 = u.cost        // cost of best path to u previously known
if(c1 < c2){       // if the path through v is better
  u.cost = c1
  u.path = v       // for computing actual paths
```

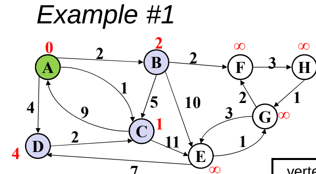

Example #1



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	
H		??	

Example #1

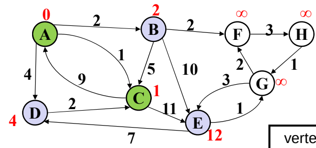


Order Added to Known Set:

A

vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C		≤ 1	A
D		≤ 4	A
E		??	
F		??	
G		??	
H		??	

Example #1

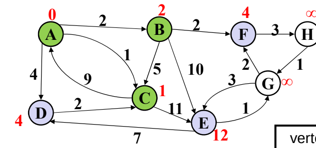


Order Added to Known Set:

A, C

vertex	known?	cost	path
A	Y	0	
B		≤ 2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		??	
G		??	
H		??	

Example #1

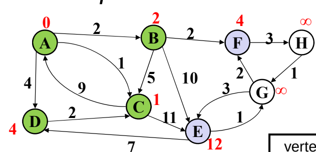


Order Added to Known Set:

A, C, B

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D		≤ 4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Example #1

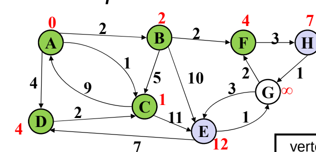


Order Added to Known Set:

A, C, B, D

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F		≤ 4	B
G		??	
H		??	

Example #1

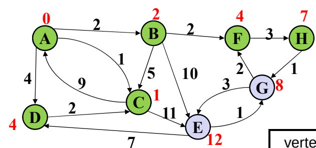


Order Added to Known Set:

A, C, B, D, F

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		??	
H		≤ 7	F

Example #1

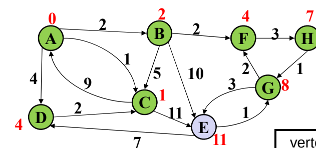


Order Added to Known Set:

A, C, B, D, F, H

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 12	C
F	Y	4	B
G		≤ 8	H
H	Y	7	F

Example #1

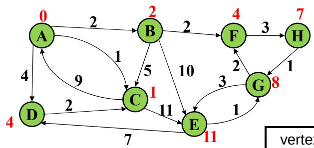


Order Added to Known Set:

A, C, B, D, F, H, G

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E		≤ 11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Example #1



Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

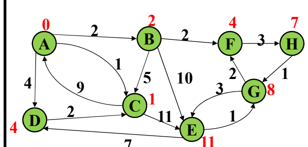
Features

- When a vertex is marked known, the cost of the shortest path to that node is known
  – The path is also known by following back-pointers
- While a vertex is still not known, another shorter path to it might still be found

Note: The "Order Added to Known Set" is not important
– A detail about how the algorithm works (client doesn't care)
– Not used by the algorithm (implementation doesn't care)
– It is sorted by path-cost, resolving ties in some way
  • Helps give intuition of why the algorithm works

Interpreting the Results

- Now that we're done, how do we get the path from, say, A to E?



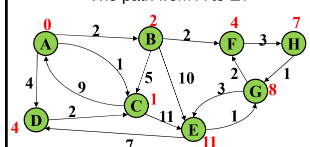
Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

Stopping Short

- How would this have worked differently if we were only interested in:
  – The path from A to G?
  – The path from A to E?

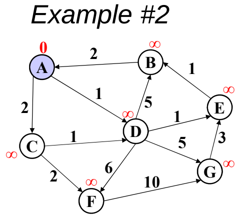


Order Added to Known Set:

A, C, B, D, F, H, G, E

vertex	known?	cost	path
A	Y	0	
B	Y	2	A
C	Y	1	A
D	Y	4	A
E	Y	11	G
F	Y	4	B
G	Y	8	H
H	Y	7	F

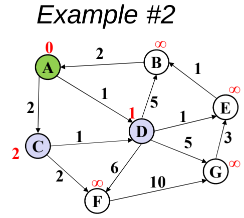
Example #2



Order Added to Known Set:

vertex	known?	cost	path
A		0	
B		??	
C		??	
D		??	
E		??	
F		??	
G		??	

Example #2

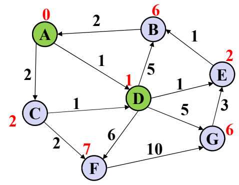


Order Added to Known Set:

A

vertex	known?	cost	path
A	Y	0	
B		??	
C		≤ 2	A
D		≤ 1	A
E		??	
F		??	
G		??	

Example #2

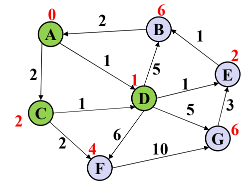


Order Added to Known Set:

A, D

vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C		≤ 2	A
D	Y	1	A
E		≤ 2	D
F		≤ 7	D
G		≤ 6	D

Example #2

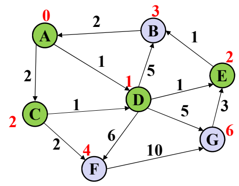


Order Added to Known Set:

A, D, C

vertex	known?	cost	path
A	Y	0	
B		≤ 6	D
C	Y	2	A
D	Y	1	A
E		≤ 2	D
F		≤ 4	C
G		≤ 6	D

Example #2

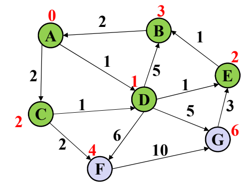


Order Added to Known Set:

A, D, C, E

vertex	known?	cost	path
A	Y	0	
B		≤ 3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Example #2

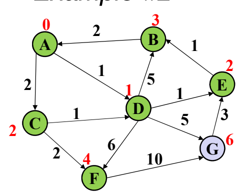


Order Added to Known Set:

A, D, C, E, B

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F		≤ 4	C
G		≤ 6	D

Example #2

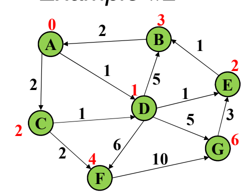


Order Added to Known Set:

A, D, C, E, B, F

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G		≤ 6	D

Example #2

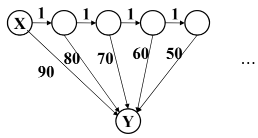


Order Added to Known Set:

A, D, C, E, B, F, G

vertex	known?	cost	path
A	Y	0	
B	Y	3	E
C	Y	2	A
D	Y	1	A
E	Y	2	D
F	Y	4	C
G	Y	6	D

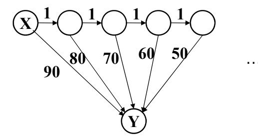
Example #3



How will the best-cost-so-far for Y proceed?

Is this expensive?

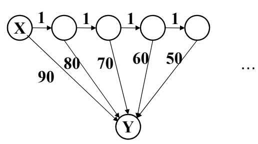
Example #3



How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, …

Is this expensive?

Example #3



How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, …

Is this expensive? No, each *edge* is processed only once

A Greedy Algorithm

- Dijkstra's algorithm
  - For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges
- An example of a *greedy algorithm*:
  - At each step, always does what seems best at that step
    - A locally optimal step, not necessarily globally optimal
  - Once a vertex is known, it is not revisited
    - Turns out to be globally optimal

Where are We?

- Had a problem: Compute shortest paths in a weighted graph with no negative weights
- Learned an algorithm: Dijkstra's algorithm
- What should we do after learning an algorithm?
  - Prove it is correct
    - Not obvious!
    - We will sketch the key ideas
  - Analyze its efficiency
    - Will do better by using a data structure we learned earlier!

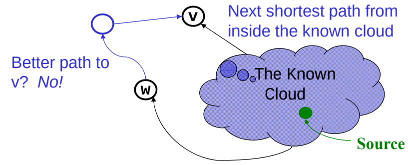
Correctness: Intuition

Rough intuition:

All the "known" vertices have the correct shortest path
- True initially: shortest path to start node has cost 0
- If it stays true every time we mark a node "known", then by induction this holds and eventually everything is "known"

Key fact we need: When we mark a vertex "known" we won't discover a shorter path later!
- This holds only because Dijkstra's algorithm picks the node with the next shortest path-so-far
- The proof is by contradiction…

Correctness: The Cloud (Rough Sketch)



Suppose **v** is the next node to be marked known ("added to the cloud")

- The best-known path to **v** must have only nodes "in the cloud"
  - Else we would have picked a node closer to the cloud than **v**
- Suppose the actual shortest path to **v** is different
  - It won't use only cloud nodes, or we would know about it
  - So it must use non-cloud nodes. Let **w** be the *first* non-cloud node on this path. The part of the path up to **w** is already known and must be shorter than the best-known path to **v**. So **v** would not have been picked. Contradiction.

Efficiency, first approach

Use pseudocode to determine asymptotic run-time

- Notice each edge is processed only once

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  while not all nodes are known:
    b = find unknown node with smallest cost
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         a.cost = b.cost + weight((b,a))
         a.path = b
```


Efficiency, first approach

Use pseudocode to determine asymptotic run-time

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```
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```

O(|V|)

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```

O(|V|)

O(|V|^2)

Efficiency, first approach

Use pseudocode to determine asymptotic run-time

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Dijkstra(V, E, vStart):
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     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         a.cost = b.cost + weight((b,a))
         a.path = b
```

O(|V|)

O(|V|^2)

O(|E|)

Efficiency, first approach

Use pseudocode to determine asymptotic run-time

- Notice each edge is processed only once

```
Dijkstra(V, E, vStart):
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       if b.cost + weight((b,a)) < a.cost:
         a.cost = b.cost + weight((b,a))
         a.path = b
```

O(|V|)

O(|V|^2)

O(|E|)

O(|V|^2)

Improving asymptotic running time

- So far: $O(|V|^2)$
- We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges
- Solution?

Improving (?) asymptotic running time

- So far: $O(|V|^2)$
- We had a similar "problem" with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges
- Solution?
  - A priority queue holding all unknown nodes, sorted by cost
  - But must support **`decreaseKey`** operation
    - Must maintain a reference from each node to its current position in the priority queue
    - Conceptually simple, but can be a pain to code up

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  build-heap with all vertices
  while heap is not empty:
    b = deleteMin()
    b.known = True
    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         decreaseKey(a,"new cost – old cost")
         a.path = b
```


Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
  for v in V:
    v.cost=infinity; v.known=False
  vStart.cost = 0
  build-heap with all vertices
  while heap is not empty:
    b = deleteMin()
    b.known = True
    for edge = (b,a) in E:
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```

O(|V|)

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
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  vStart.cost = 0
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  while heap is not empty:
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    for edge = (b,a) in E:
     if not a.known:
       if b.cost + weight((b,a)) < a.cost:
         decreaseKey(a,"new cost – old cost")
         a.path = b
```

O(|V|)

O(|V|log|V|)

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
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  vStart.cost = 0
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```

O(|V|)

O(|V|log|V|)

O(|E|log|V|)

Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```
Dijkstra(V, E, vStart):
  for v in V:
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       if b.cost + weight((b,a)) < a.cost:
         decreaseKey(a,"new cost – old cost")
         a.path = b
```

O(|V|)

O(|V|log|V|)

O(|E|log|V|)

O(|V|log|V|+|E|log|V|)

Dense vs. sparse again

- First approach: $O(|V|^2)$
- Second approach: $O(|V|\log|V|+|E|\log|V|)$
- So which is better?
  - Sparse: $O(|V|\log|V|+|E|\log|V|)$ (if $|E| > |V|$, then $O(|E|\log|V|)$)
  - Dense: $O(|V|^2)$
- But, remember these are worst-case and asymptotic
  - Priority queue might have slightly worse constant factors
  - On the other hand, for "normal graphs", we might call **`decreaseKey`** rarely (or not percolate far), making $|E|\log|V|$ more like $|E|$