CSE373: Data Structures and Algorithms

Minimum Spanning Trees and Kruskal's Algorithm

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This lecture material represents the work of multiple instructors at the University of Washington. Thank you to all who have contributed!

Minimum Spanning Trees

The minimum-spanning-tree problem

- Given a weighted undirected graph, compute a spanning tree of minimum weight

Given an undirected graph G=(V,E), find a graph G'=(V, E') such that:

- E' is a subset of E
- |E'| = |V| - 1
- G' is connected

G' is a minimum spanning tree.

Minimum Spanning Tree Algorithms

- Kruskal's Algorithm for Minimum Spanning Tree construction
  - A greedy algorithm.
  - Uses a priority queue.
  - Uses the UNION-FIND technique.

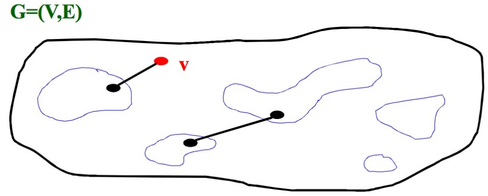
- Prim's Algorithm for Minimum Spanning Tree
  - Related to Dijkstra's Algorithm for shortest paths.
  - Both based on expanding cloud of known vertices (basically using a priority queue instead of a DFS stack)

Kruskal's Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

An *edge-based* greedy algorithm
Builds MST by greedily adding edges

G=(V,E)

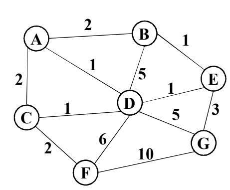


Kruskal's Algorithm Pseudocode

1. Sort edges by weight (better: put in min-heap)
2. Put each node in its own subset (of a UNION-FIND instance).
3. While output size **< |V|-1**
   - Consider next smallest edge `(u,v)`
   - if `find(u)` and `find(v)` indicate `u` and `v` are in different sets
     - output `(u,v)`
     - Perform `union(find(u),find(v))`

Recall invariant:
`u` and `v` in same set if and only if connected in output-so-far

Kruskal's Example

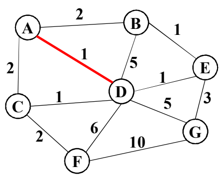


Edges in sorted order:
1: (A,D), (C,D), (B,E), (D,E)
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output:

Note: At each step, the UNION-FIND subsets correspond to the trees in a forest.

Kruskal's Example



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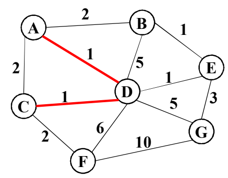
6: (D,F)
10: (F,G)

Output: (A,D)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:
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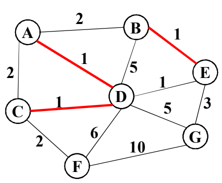
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



Edges in sorted order:
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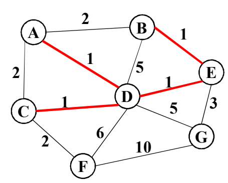
2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D), (B,E)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



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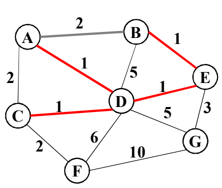
Output: (A,D), (C,D), (B,E), (D,E)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



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Output: (A,D), (C,D), (B,E), (D,E)

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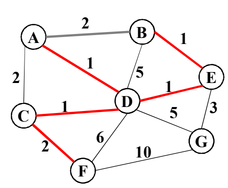
Kruskal's Example



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2: (A,B), (C,F), (A,C)
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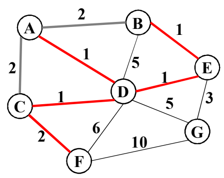
6: (D,F)
10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



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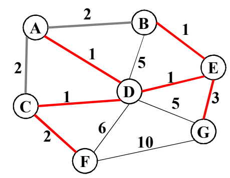
Output: (A,D), (C,D), (B,E), (D,E), (C,F)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Example



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2: (A,B), (C,F), (A,C)
3: (E,G)
5: (D,G), (B,D)
6: (D,F)
10: (F,G)

Output: (A,D), (C,D), (B,E), (D,E), (C,F), (E,G)

Note: At each step, the union/find sets are the trees in the forest

Kruskal's Algorithm Analysis

Idea: Grow a forest out of edges that do not grow a cycle. (This is similar to the maze-construction problem: knocking down a wall was essentially adding an edge that connected adjacent cells.)

- But now consider the edges in order by weight

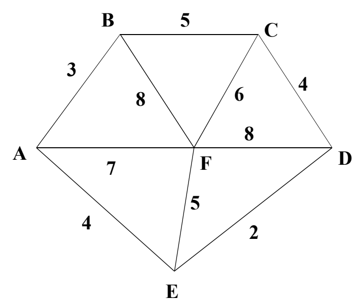
So:

- Sort edges: $O(|\mathbf{E}| \log |\mathbf{E}|)$
- Iterate through edges using union-find for cycle detection almost $O(|\mathbf{E}|)$

Somewhat better:

- Floyd's algorithm to build min-heap with edges $O(|\mathbf{E}|)$
- Iterate through edges using UNION-FIND for cycle prevention and `deleteMin` to get next edge $O(|\mathbf{E}| \log |\mathbf{E}|)$
- Not better *worst-case* asymptotically, but often stops long before considering all edges.

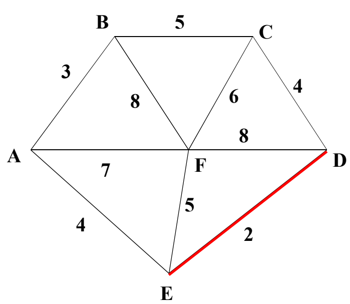
Kruskal's Algorithm



List the edges in order of size:

ED 2
AB 3
AE 4
CD 4
BC 5
EF 5
CF 6
AF 7
BF 8
CF 8

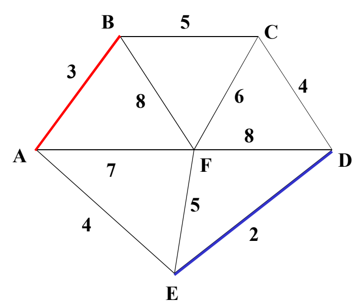
Kruskal's Algorithm



Select the edge with min cost

ED 2

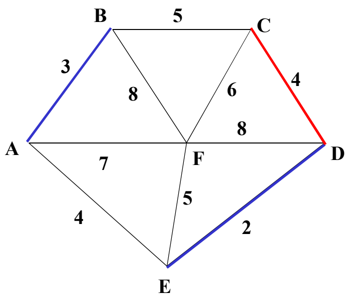
Kruskal's Algorithm



Select the next minimum cost edge that does not create a cycle

ED 2
AB 3

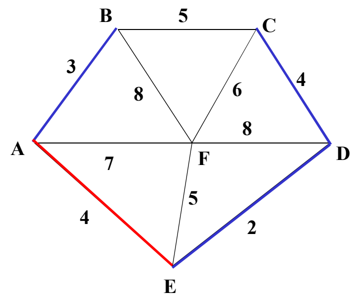
Kruskal's Algorithm



Select the next minimum cost edge that does not create a cycle

ED 2
AB 3
CD 4 (or AE 4)

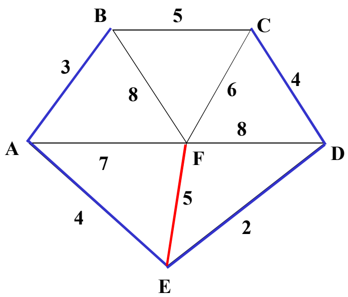
Kruskal's Algorithm



Select the next minimum cost edge that does not create a cycle

ED 2
AB 3
CD 4
AE 4

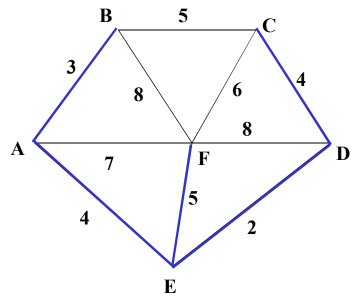
Kruskal's Algorithm



Select the next minimum cost edge that does not create a cycle

ED 2
AB 3
CD 4
AE 4
BC 5 – forms a cycle
EF 5

Kruskal's Algorithm



All vertices have been connected.

The solution is

ED 2
AB 3
CD 4
AE 4
EF 5

Total weight of tree: 18