CSE373: Data Structures and Algorithms

Priority Queues and Binary Heaps

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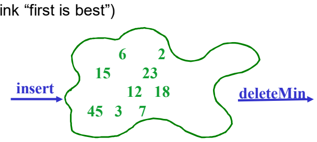
This lecture material represents the work of multiple instructors at the University of Washington.
Thank you to all who have contributed!

A new ADT: Priority Queue

- A **priority queue** holds *compare-able data*
  - Like dictionaries, we need to *compare items*
    - Given *x* and *y*, is *x* less than, equal to, or greater than *y*
    - Meaning of the ordering can depend on your data
  - Integers are comparable, so will use them in examples
    - But the priority queue ADT is much more general
    - Typically two fields, the *priority* and the *data*

Priorities

- Each item has a "priority"
  - In our examples, the *lesser* item is the one with the *greater* priority
  - So "priority 1" is more important than "priority 4"
  - (Just a convention, think "first is best")
- Operations:
  - `insert`
  - `deleteMin`
  - `is_empty`



- Key property: `deleteMin` *returns* and *deletes* the item with greatest priority (lowest priority value)
  - Can resolve ties arbitrarily

Example

`insert` *x1* with priority *5*	*(x1,5)*
`insert` *x2* with priority *3*	*(x1,5) (x2,3)*
`insert` *x3* with priority *4*	*(x1,5) (x3,4) (x2,3)*
a = `deleteMin // x2`	*(x1,5) (x3,4)*
b = `deleteMin // x3`	*(x1,5)*
`insert` *x4* with priority *2*	
`insert` *x5* with priority *6*	
c = `deleteMin // x4`	
d = `deleteMin // x1`	

- Analogy: `insert` is like `enqueue`, `deleteMin` is like `dequeue`
  - But the whole point is to use priorities instead of FIFO

Applications

Like all good ADTs, the priority queue arises often
- Sometimes blatant, sometimes less obvious

- Run multiple programs in the operating system
  - "critical" before "interactive" before "compute-intensive"
  - Maybe let users set priority level
- Treat hospital patients in order of severity (or triage)
- Select print jobs in order of decreasing length?
- Forward network packets in order of urgency
- Select most frequent symbols for data compression
- Sort (first `insert` all, then repeatedly `deleteMin`)

Finding a good data structure

- Will show an efficient, non-obvious data structure for this ADT
  - But first let's analyze some "obvious" ideas for *n* data items
  - All times worst-case; assume arrays "have room"

data	*insert algorithm / time*		*deleteMin algorithm / time*	
unsorted array	add at end	$O(1)$	search	$O(n)$
unsorted linked list	add at front	$O(1)$	search	$O(n)$
sorted circular array	search / shift	$O(n)$	move front	$O(1)$
sorted linked list	put in right place	$O(n)$	remove at front	$O(1)$
binary search tree	put in right place	$O(n)$	leftmost	$O(n)$
AVL tree	put in right place	$O(\log n)$	leftmost	$O(\log n)$

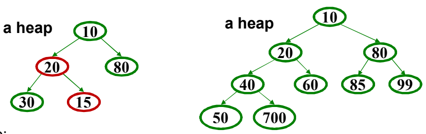
Our data structure: the Binary Heap

A *binary min-heap* (or just *binary heap* or just *heap*) has:

- Structure property: A *complete* binary tree
- Heap property: The priority of every (non-root) node is less than the priority of its parent
  - ***Not** a binary search tree*

not a heap

a heap

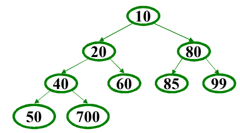


So:

- Where is the most important item?
- What is the height of a heap with *n* items?

Operations: basic idea

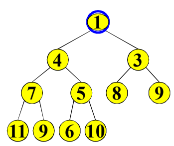
- `deleteMin`:
  1. Remove root node
  2. Move right-most node in last row to root to restore structure property
  3. "Percolate down" to restore heap property
- `insert`:
  1. Put new node in next position on bottom row to restore structure property
  2. "Percolate up" to restore heap property



Overall strategy:
- *Preserve structure property*
- *Break and restore heap property*

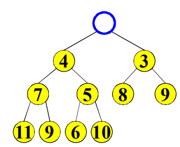
`deleteMin`

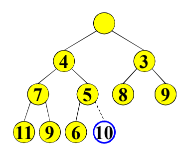
Delete (and later return) value at root node



`deleteMin`*: Keep the Structure Property*

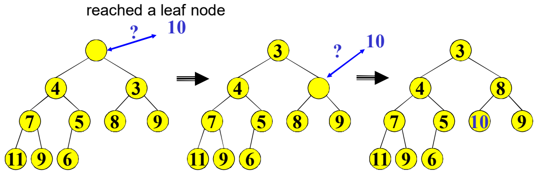
- We now have a "hole" at the root
  - Need to fill the hole with another value
- Keep structure property: When we are done, the tree will have one less node and must still be complete
- Pick the last node on the bottom row of the tree and move it to the "hole"





`deleteMin`*: Restore the Heap Property*

Percolate down:
- Keep comparing priority of item with both children
- If priority is less important, swap with the most important child and go down one level
- Done if both children are less important than the item or we've reached a leaf node



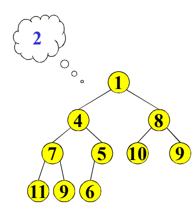
Run time?

Runtime is O(height of heap) $O(\log n)$

Height of a complete binary tree of *n* nodes = $\lfloor \log_2(n) \rfloor$

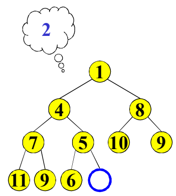
`insert`

- Add a value to the tree
- Afterwards, structure and heap properties must still be correct



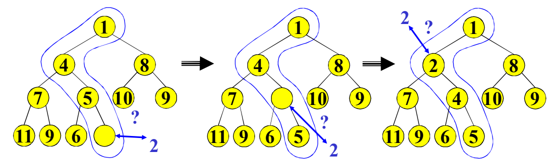
`insert`: Maintain the Structure Property

- There is only one valid tree shape after we add one more node
- So put our new data there and then focus on restoring the heap property



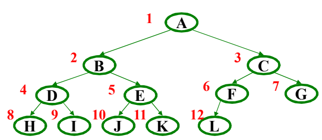
`insert`: Restore the heap property

Percolate up:
- Put new data in new location
- If parent is less important, swap with parent, and continue
- Done if parent is more important than item or reached root



What is the running time?
Like `deleteMin`, worst-case time proportional to tree height: $O(\log n)$

Array Representation of Binary Trees



From node `i`:

left child: `i*2`
right child: `i*2+1`
parent: `i/2`

(wasting index 0 is convenient for the index arithmetic)

implicit (array) implementation:

	A	B	C	D	E	F	G	H	I	J	K	L	
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Judging the array implementation

Plusses:
- Non-data space: just index 0 and unused space on right
  - In conventional tree representation, one edge per node (except for root), so n-1 wasted space (like linked lists)
  - Array would waste more space if tree were not complete
- Multiplying and dividing by 2 is very fast (shift operations in hardware)
- Last used position is just index `size`

Minuses:
- Same might-be-empty or might-get-full problems we saw with stacks and queues (resize by doubling as necessary)

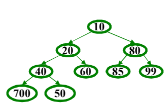
Plusses outweigh minuses: "this is how people do it"

This pseudocode uses ints. In real use, you will have data nodes with priorities.

Pseudocode: insert into binary heap

```
void insert(int val) {
  if(size==arr.length-1)
    resize();
  size++;
  i=percolateUp(size,val);
  arr[i] = val;
}
```

```
int percolateUp(int hole,
                int val) {
  while(hole > 1 &&
        val < arr[hole/2])
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```



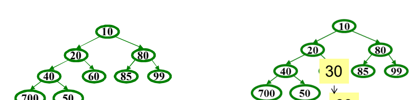
	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

This pseudocode uses ints. In real use, you will have data nodes with priorities.

insert(30) percolateUp(10, 30)

```
void insert(int val) {
  if(size==arr.length-1)
    resize();
  size++;
  i=percolateUp(size,val);
  arr[i] = val;
}
```

```
int percolateUp(int hole,
                int val) {
  while(hole > 1 &&
        val < arr[hole/2])
    arr[hole] = arr[hole/2];
    hole = hole / 2;
  }
  return hole;
}
```



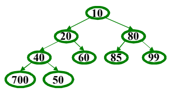
30 30

	10	20	80	40		85	99	700	50	60			
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Pseudocode: `deleteMin` from binary heap

```
int deleteMin() {
  if(isEmpty()) throw…
  ans = arr[1];
  hole = percolateDown
           (1,arr[size]);
  arr[hole] = arr[size];
  size--;
  return ans;
}
```

```
int percolateDown(int hole,
                  int val) {
 while(2*hole <= size) {
  left  = 2*hole;
  right = left + 1;
  if(right > size ||
     arr[left] < arr[right])
    target = left;
  else
    target = right;
  if(arr[target] < val) {
    arr[hole] = arr[target];
    hole = target;
  } else
      break;
 }
 return hole;
}
```

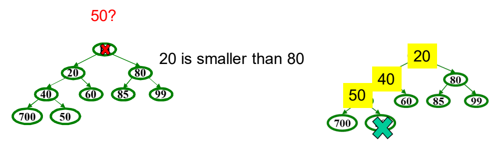


	10	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13

Pseudocode: `deleteMin` from binary heap

20 40 50

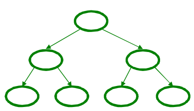
	1	20	80	40	60	85	99	700	50				
0	1	2	3	4	5	6	7	8	9	10	11	12	13



Example

1. `insert`: 16, 32, 4, 67, 105, 43, 2
2. `deleteMin`

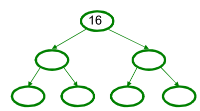
0	1	2	3	4	5	6	7



Example

1. `insert`: 16, 32, 4, 67, 105, 43, 2
2. `deleteMin`

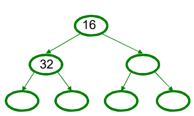
	16						
0	1	2	3	4	5	6	7



Example

1. `insert`: 16, 32, 4, 67, 105, 43, 2
2. `deleteMin`

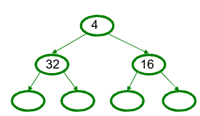
	16	32					
0	1	2	3	4	5	6	7



Example

1. `insert`: 16, 32, 4, 67, 105, 43, 2
2. `deleteMin`

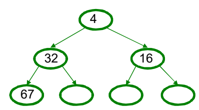
	4	32	16				
0	1	2	3	4	5	6	7



Example

1. **insert**: 16, 32, 4, 67, 105, 43, 2
2. **deleteMin**

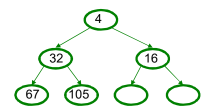
	4	32	16	67			
0	1	2	3	4	5	6	7



Example

1. **insert**: 16, 32, 4, 67, 105, 43, 2
2. **deleteMin**

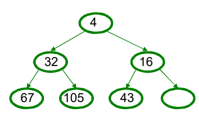
	4	32	16	67	105		
0	1	2	3	4	5	6	7



Example

1. **insert**: 16, 32, 4, 67, 105, 43, 2
2. **deleteMin**

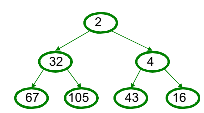
	4	32	16	67	105	43	
0	1	2	3	4	5	6	7



Exercise

1. **insert**: 16, 32, 4, 67, 105, 43, 2
2. **deleteMin**

	2	32	4	67	105	43	16
0	1	2	3	4	5	6	7



Other operations

- **decreaseKey**: given pointer to object in priority queue (e.g., its array index), lower its priority value by *p*
  - Change priority and percolate up
- **increaseKey**: given pointer to object in priority queue (e.g., its array index), raise its priority value by *p*
  - Change priority and percolate down
- **remove**: given pointer to object in priority queue (e.g., its array index), remove it from the queue
  - **decreaseKey** with $p = \infty$, then **deleteMin**

Running time for all these operations?

Build Heap

- Suppose you have *n* items to put in a new (empty) priority queue
  - Call this operation **buildHeap**
- *n* **inserts** works
  - Only choice if ADT doesn't provide **buildHeap** explicitly
  - $O(n \log n)$
- Why would an ADT provide this unnecessary operation?
  - Convenience
  - Efficiency: an $O(n)$ algorithm called Floyd's Method
  - Common issue in ADT design: how many specialized operations

Floyd's Method

1. Use *n* items to make any complete tree you want
   – That is, put them in array indices 1,…,*n*

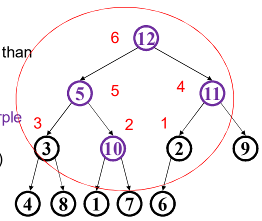
2. Treat it as a heap and fix the heap-order property
   – Bottom-up: leaves are already in heap order, work up toward the root one level at a time

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

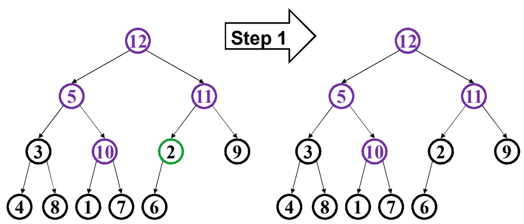

Example

12/2 = 6

- In tree form for readability
  – Purple for node not less than descendants
    • heap-order problem
  – Notice no leaves are purple
  – Check/fix each non-leaf bottom-up (6 steps here)

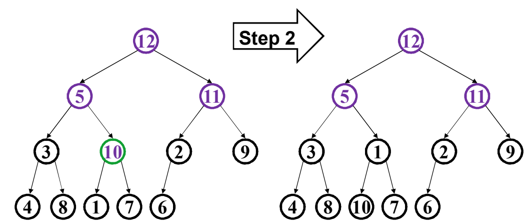


Example



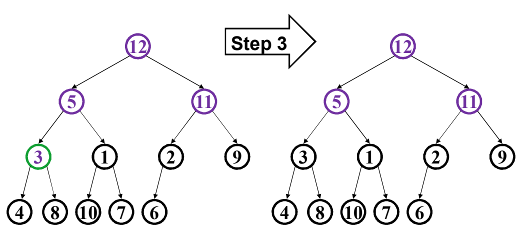
- Happens to already be less than children (er, child)

Example



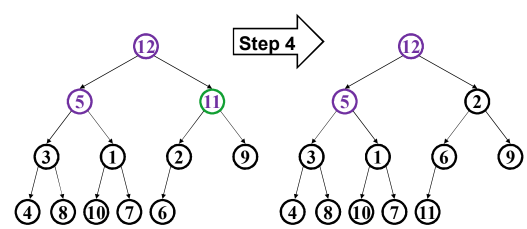
- Percolate down (notice that moves 1 up)

Example



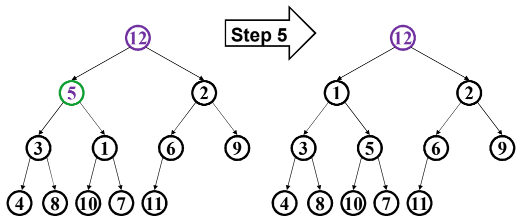
- Another nothing-to-do step

Example

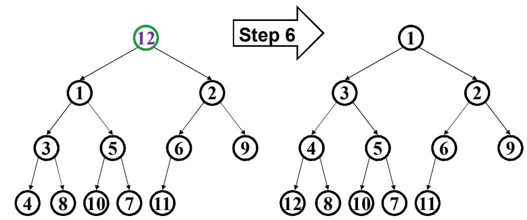


- Percolate down as necessary (steps 4a and 4b)

Example



Example



But is it right?

- "Seems to work"
  - Let's *prove* it restores the heap property (correctness)
  - Then let's *prove* its running time (efficiency)

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```


Correctness

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Loop Invariant: For all `j>i`, `arr[j]` is less than its children

- True initially: If `j > size/2`, then `j` is a leaf
  - Otherwise its left child would be at position > `size`
- True after one more iteration: loop body and `percolateDown` make `arr[i]` less than children without breaking the property for any descendants

So after the loop finishes, all nodes are less than their children

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Easy argument: `buildHeap` is *O*(*n* `log` *n*) where *n* is `size`

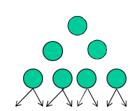
- `size/2` loop iterations
- Each iteration does one `percolateDown`, each is *O*(`log` *n*)

This is correct, but there is a more precise ("tighter") analysis of the algorithm…

Efficiency

```
void buildHeap() {
  for(i = size/2; i>0; i--) {
    val  = arr[i];
    hole = percolateDown(i,val);
    arr[hole] = val;
  }
}
```

Better argument: `buildHeap` is *O*(*n*) where *n* is `size`



- `size/2` total loop iterations: *O*(*n*)
- 1/2 the loop iterations percolate at most 1 step
- 1/4 the loop iterations percolate at most 2 steps
- 1/8 the loop iterations percolate at most 3 steps
- …
- ((1/2) + (2/4) + (3/8) + (4/16) + (5/32) + …) < 2 (page 4 of Weiss)
  - So at most `2*(size/2)` *total* percolate steps: *O*(*n*)

Lessons from **`buildHeap`**

- Without **`buildHeap`**, our ADT already let clients implement their own in $O(n \log n)$ worst case
- By providing a specialized operation internal to the data structure (with access to the internal data), we can do $O(n)$ worst case
  - Intuition: Most data is near a leaf, so better to percolate down
- Can analyze this algorithm for:
  - Correctness:
    - Non-trivial inductive proof using loop invariant
  - Efficiency:
    - First analysis easily proved it was $O(n \log n)$
    - Tighter analysis shows same algorithm is $O(n)$

Exercise: Build the Heap using Floyd's method

3	9	2	6	25	1	80	35
1	2	3	4	5	6	7	8