CSE373: Data Structures and Algorithms

Implementing the UNION-FIND ADT

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This lecture material represents the work of multiple instructors at the University of Washington. Thank you to all who have contributed!

The plan

Last lecture:

- Disjoint sets
- The UNION-FIND ADT for disjoint sets

Today's lecture:

- Basic implementation of the UNION-FIND ADT with "up trees"
- Optimizations that make the implementation much faster

Union-Find ADT

- Given an unchanging set *S*, **`create`** an initial partition of a set
  - Typically each item in its own subset: {a}, {b}, {c}, …
  - Give each subset a "name" by choosing a *representative element*
- Operation **`find`** takes an element of *S* and returns the representative element of the subset it is in
- Operation **`union`** takes two subsets and (permanently) makes one larger subset
  - A different partition with one fewer set
  - Affects result of subsequent **`find`** operations
  - Choice of representative element up to implementation

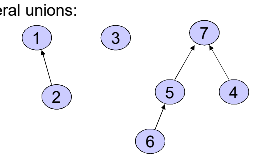
Implementation – our goal

- Start with an initial partition of *n* subsets
  - Often 1-element sets, e.g., {1}, {2}, {3}, …, {*n*}
- May have *m* **`find`** operations
- May have up to *n*-1 **`union`** operations in any order
  - After *n*-1 **`union`** operations, every **`find`** returns same 1 set

Up-tree data structure

- Tree with:
  - No limit on branching factor
  - References from children to parent
- Start with *forest* of 1-node trees

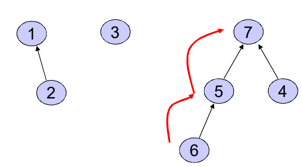
- Possible forest after several unions:
  - Will use roots for set names



Find

`find(x)`:
- *Assume* we have *O*(1) access to each node
  - Will use an array where index **`i`** holds node **`i`**
- Start at **`x`** and follow parent pointers to root
- Return the root

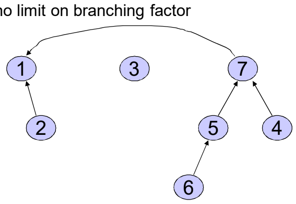
`find`(6) = 7



Union

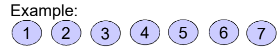
`union(x,y)`:
- Assume `x` and `y` are roots
  - Else `find` the roots of their trees
- Assume distinct trees (else do nothing)
- Change root of one to have parent be the root of the other
  - Notice no limit on branching factor

`union`(1,7)



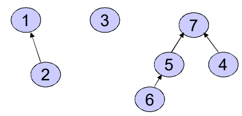
Simple implementation

- If set elements are contiguous numbers (e.g., 1,2,…,*n*), use an array of length *n* called `up`
  - Starting at index 1 on slides
  - Put in array index of parent, with 0 (or -1, etc.) for a root
- Example:



	1	2	3	4	5	6	7
up	0	0	0	0	0	0	0

- Example:



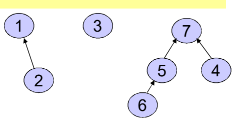
	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0

- If set elements are not contiguous numbers, could have a separate dictionary to map elements (keys) to numbers (values)

Implement operations

```
// assumes x in range 1,n
int find(int x) {
  while(up[x] != 0) {
    x = up[x];
  }
  return x;
}
```

```
// assumes x,y are roots
void union(int x, int y){
  up[y] = x;
}
```



	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0

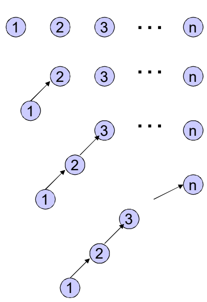
- Worst-case run-time for `union`? $O(1)$
- Worst-case run-time for `find`? $\Theta(n)$
- Worst-case run-time for *m* `find`s and *n*-1 `union`s? $\Theta(m*n)$

Two key optimizations

1. Improve `union` so it stays *O(1)* but makes `find` *O*(`log` *n*)
   - So *m* `find`s and *n*-1 `union`s is in *O*(*m* `log` *n* + *n*)
   - *Union-by-size:* connect smaller tree to larger tree

2. Improve `find` so it becomes even faster
   - Make *m* `find`s and *n*-1 `union`s ***almost*** in *O*(*m* + *n*)
   - *Path-compression:* connect directly to root during finds

The bad case to avoid



`union`(2,1)

`union`(3,2)

:

.

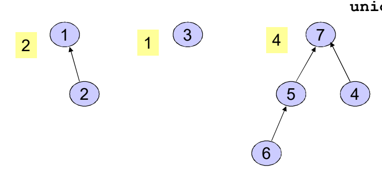
`union`(*n*,*n*-1)

`find`(1) = *n* steps!!

Union-by-size

Union-by-size:
- Always point the *smaller* (total # of nodes) tree to the root of the larger tree

`union`(1,7)

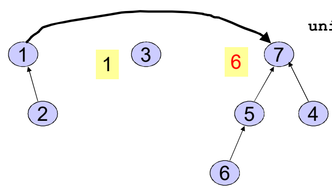


Union-by-size

Union-by-size:

– Always point the *smaller* (total # of nodes) tree to the root of the larger tree

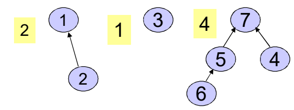
union(1,7)



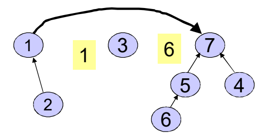
Array implementation

Keep the size (number of nodes in a second array)

– Or have one array of objects with two fields



	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0
weight	2		1				4



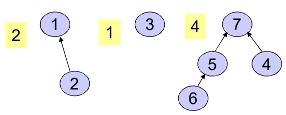
	1	2	3	4	5	6	7
up	7	1	0	7	7	5	0
weight			1				6

Nice trick

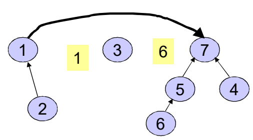
Actually we do not need a second array…

– Instead of storing 0 for a root, store negation of size

– So up value < 0 means a root

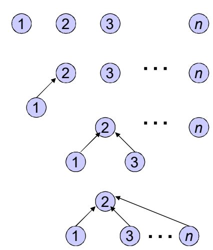


	1	2	3	4	5	6	7
up	-2	1	-1	7	7	5	-4



	1	2	3	4	5	6	7
up	7	1	-1	7	7	5	-6

The Bad case? Now a Great case…



union(2,1)

union(3,2)

:

union(*n*,*n*-1)

find(1) *constant here*

General analysis

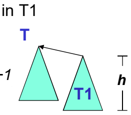
- Showing one worst-case example is now good is *not* a proof that the worst-case has improved
- So let's prove:
  - **union** is still $O(1)$ – this is "obvious"
  - **find** is now $O(\log n)$
- Claim: If we use union-by-size, an up-tree of height h has at least 2^h nodes
  - Proof by induction on *h*…

Exponential number of nodes

P(*h*)= With union-by-size, up-tree of height *h* has at least 2^h nodes

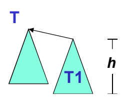
Proof by induction on *h*…

- Base case: $h = 0$: The up-tree has 1 node and $2^0 = 1$
- Inductive case: Assume P(*h*) and show P(*h*+1)
  - A height *h*+1 tree T has at least one height *h* child T1
  - T1 has at least 2^h nodes by induction
  - And T has *at least* as many nodes not in T1 than in T1
    - Else union-by-size would have had T point to T1, not T1 point to T (!!)
  - So total number of nodes is *at least* $2^h + 2^h = 2^{h+1}$



The key idea

Intuition behind the proof: No one child can have more than half the nodes

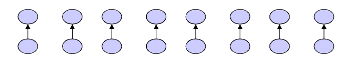


So, as usual, if number of nodes is exponential in height, then height is logarithmic in number of nodes

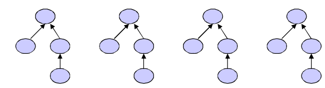
So `find` is $O($`log` $n)$

The new worst case

n/2 Unions-by-size

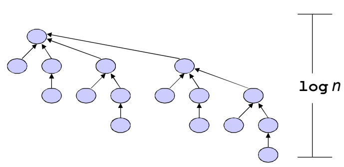


n/4 Unions-by-size



The new worst case (continued)

After n/2 + n/4 + …+ 1 Unions-by-size:



Worst find

Height grows by 1 a total of `log` n times

What about union-by-height

We could store the height of each root rather than size

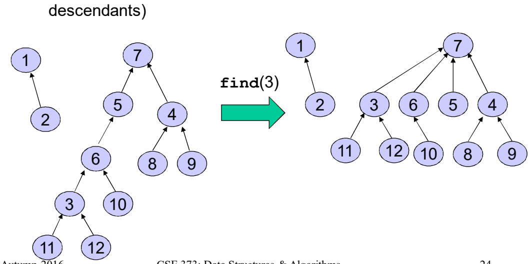
- Still guarantees logarithmic worst-case find
  - Proof left as an exercise if interested
- But does not work well with our next optimization
  - Maintaining height becomes inefficient, but maintaining size still easy

Two key optimizations

1. Improve `union` so it stays *O(1)* but makes `find` $O($`log` $n)$
   - So m `finds` and n-1 `unions` is $O(m$ `log` $n + n)$
   - *Union-by-size:* connect smaller tree to larger tree
2. Improve `find` so it becomes even faster
   - Make m `finds` and n-1 `unions` ***almost*** $O(m + n)$
   - *Path-compression:* connect directly to root during finds

Path compression

- Simple idea: As part of a `find`, change each encountered node's parent to point directly to root
  - Faster future `finds` for everything on the path (and their descendants)

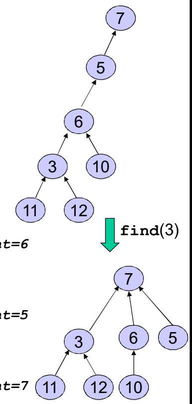


Pseudocode

```
// performs path compression
int find(i) {
  // find root
  int r = i
  while(up[r] > 0)
     r = up[r]

  // compress path
  if i==r
    return r;
  int old_parent = up[i]
  while(old_parent != r) {
    up[i] = r
    i = old_parent;
    old_parent = up[i]
  }
  return r;
}
```

Example
i=3
r=3
r=6
r=5
r=7
old_parent=6
up[3]=7
i=6
old_parent=5
up[6]=7
i=5
old_parent=7



So, how fast is it?

A single worst-case **find** could be $O(\log n)$
- But only if we did a lot of worst-case unions beforehand
- And path compression will make future finds faster

Turns out the amortized worst-case bound is much better than $O(\log n)$
- We won't *prove* it – see text if curious
- But we will *understand* it:
  - How it is *almost* $O(1)$
  - Because total for *m* **finds** and *n*-1 **unions** is *almost* $O(m+n)$

A really slow-growing function

The "log star" function:

log* *x* is the minimum number of times you need to apply "**log** of **log** of **log** of" to go from *x* to a number <= 1

For just about every number we care about, **log*** *x* is *5* or less!
If $x \leq 2^{65536}$ then **log*** $x \leq 5$
- $\log^* 2 = 1$
- $\log^* 4 = \log^* 2^2 = 2$
- $\log^* 16 = \log^* 2^{(2^2)} = 3$ (**log log log** 16 = 1)
- $\log^* * 65536 = \log^* 2^{((2^2)^2)} = 4$ (**log log log log** 65536 = 1)
- $\log^* 2^{65536} = \ldots\ldots\ldots\ldots\ldots = 5$

Almost linear

- Turns out total time for *m* **finds** and *n*-1 **unions** is
  $O((m+n) \cdot (\log^* (m+n))$
  - Remember, if $m+n < 2^{65536}$ then $\log^* (m+n) < 5$ so effectively we have $O(m+n)$
- Because **log*** grows soooo slowly
  - For all practical purposes, amortized bound is constant, i.e., cost of find is constant, total cost for m finds is linear
  - We say "near linear" or "effectively linear"
- Need union-by-size and path-compression for this bound
  - Path-compression changes height but not weight, so they interact well
- As always, asymptotic analysis is separate from "coding it up"

Curious about the Proof?

See the textbook!