CSE373: Data Structures and Algorithms

Disjoint Sets and the UNION-FIND ADT

Steve Tanimoto
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This lecture material represents the work of multiple instructors at the University of Washington. Thank you to all who have contributed!

Where we are

Last lecture:
- Hashing and collision resolution

Today:
- Disjoint sets
- The UNION-FIND ADT for disjoint sets

Next lecture:
- Basic implementation of the UNION-FIND ADT with "up trees"
- Optimizations that make the implementation much faster

Disjoint sets

- A set is a collection of elements (no-repeats)
- In computer science, two sets are said to be disjoint if they have no element in common.
  - $\mathbf{S_1 \cap S_2 = \varnothing}$
- For example, {1, 2, 3} and {4, 5, 6} are disjoint sets.
- For example, {x, y, z} and {t, u, x} are not disjoint.

Partitions

A partition *P* of a set *S* is a set of sets $\{S_1, S_2, \ldots, S_n\}$ such that every element of *S* is in **exactly one** S_i .

Put another way:

$S_1 \cup S_2 \cup \ldots \cup S_k = S$

$i \neq j$ implies $S_i \cap S_j = \varnothing$ (sets are *pairwise disjoint*)

Example:
- Let *S* be {a,b,c,d,e}
- One partition: {a}, {d,e}, {b,c}
- Another partition: {a,b,c}, ∅, {d}, {e}
- A third: {a,b,c,d,e}
- Not a partition: {a,b,d}, {c,d,e} *…. element d appears twice*
- Not a partition of *S*: {a,b}, {e,c} *…. missing element d*

Binary relations

- *S x S* is the set of all pairs of elements of *S* (cartesian product)
  - Example: If *S* = {a,b,c} then *S x S* = {(a,a),(a,b),(a,c),(b,a),(b,b),(b,c), (c,a),(c,b),(c,c)}
- A binary relation *R* on a set *S* is any subset of *S x S*
  - i.e., a collection of ordered pairs of elements of S.
  - Write *R*(x,y) to mean (x,y) is in the relation.
  - (Unary, ternary, quaternary, … relations defined similarly)
- Examples for *S* = people-in-this-room
  - Sitting-next-to-each-other relation
  - First-sitting-right-of-second relation
  - Went-to-same-high-school relation
  - First-is-younger-than-second relation

Properties of binary relations

- A relation *R* over set *S* is reflexive means *R*(x, x) for *all* x in *S*
  - e.g., The relation "≤" on the set of integers {1, 2, 3} is {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)}
    It is reflexive because (1, 1), (2, 2), (3, 3) are in this relation.
- A relation R on a set S is symmetric if and only if for any x and y in S, whenever (x, y) is in R , (y, x) is in R .
  - e.g., The relation "=" on the set of integers {1, 2, 3} is {(1, 1) , (2, 2) (3, 3) } and it is symmetric.
  - The relation "being acquainted with" on a set of people is symmetric.
- A binary relation *R* over set *S* is transitive means:
  If *R*(x, y) and *R*(y, z) then *R*(x, z) for *all* a,b,c in *S*
  - e.g., The relation "≤" on the set of integers {1, 2, 3} is transitive, because for (1, 2) and (2, 3) in "≤", (1, 3) is also in "≤" (and similarly for the others)

Equivalence relations

- A binary relation *R* is an equivalence relation if *R* is reflexive, symmetric, *and* transitive
- Examples
  - Same gender
  - Connected roads in the world
  - "Is equal to" on the set of real numbers
  - "Has the same birthday as" on the set of all people
  - …

Punch-line

- Equivalence relations give rise to partitions.
- Every partition induces an equivalence relation
- Every equivalence relation induces a partition
- Suppose $P = \{S_1, S_2, \ldots, S_n\}$ is a partition
  - Define $R(x,y)$ to mean x and y are in the same S_i
    - *R* is an equivalence relation
- Suppose *R* is an equivalence relation over *S*
  - Consider a set of sets $S_1, S_2, \ldots, S_n$ where
    (1) x and y are in the same S_i if and only if R(x,y)
    (2) Every x is in some S_i
    - This set of sets is a partition

Example

- Let *S* be {a,b,c,d,e}
- One partition: {a,b,c}, {d}, {e}
- The corresponding equivalence relation:
  (a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (d,d), (e,e)

The Union-Find ADT

- The union-find ADT (or "Disjoint Sets" or "Dynamic Equivalence Relation") keeps track of a set of elements partitioned into a number of disjoint subsets.
- Many uses (which is why an ADT taught in CSE 373):
  - Road/network/graph connectivity (will see this again)
    - "connected components" e.g., in social network
  - Partition an image by connected-pixels-of-similar-color
  - Type inference in programming languages
- Not as common as dictionaries, queues, and stacks, but valuable because implementations are very fast, so when applicable can provide big improvements

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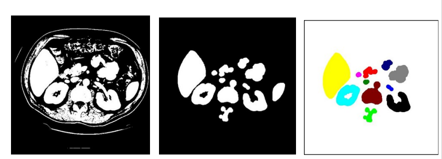
Connected Components of an Image



gray tone image binary image cleaned up components

Union-Find Operations

- Given an unchanging set *S*, `create` an initial partition of a set
  - Typically each item in its own subset: {a}, {b}, {c}, …
  - Give each subset a "name" by choosing a *representative element*
- Operation `find` takes an element of *S* and returns the representative element of the subset it is in
- Operation `union` takes two subsets and (permanently) makes one larger subset
  - A different partition with one fewer set
  - Affects result of subsequent `find` operations
  - Choice of representative element up to implementation

Example

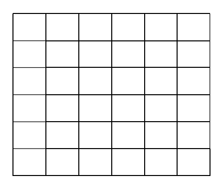
- Let *S* = {1,2,3,4,5,6,7,8,9}
- Let initial partition be (will highlight representative elements red)
  {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}
- `union`(2,5):
  {1}, {2, 5}, {3}, {4}, {6}, {7}, {8}, {9}
- `find`(4) = 4, `find`(2) = 2, `find`(5) = 2
- `union`(4,6), `union`(2,7)
  {1}, {2, 5, 7}, {3}, {4, 6}, {8}, {9}
- `find`(4) = 6, `find`(2) = 2, `find`(5) = 2
- `union`(2,6)
  {1}, {2, 4, 5, 6, 7}, {3}, {8}, {9}

No other operations

- All that can "happen" is sets get unioned
  - No "un-union" or "create new set" or …
- As always: trade-offs
  - Implementations will exploit this small ADT
- Surprisingly useful ADT
  - But not as common as dictionaries or priority queues

Example application: maze-building

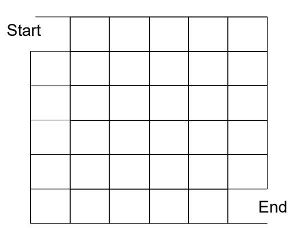
- Build a random maze by erasing edges



  - Possible to get from anywhere to anywhere
    - Including "start" to "finish"
  - No loops possible without backtracking
    - After a "bad turn" have to "undo"

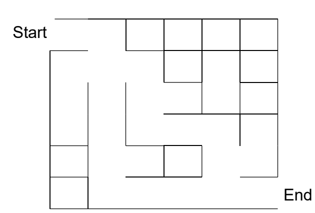
Maze building

Pick start edge and end edge



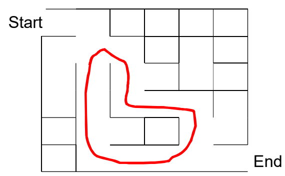
Repeatedly pick random edges to delete

One approach: just keep deleting random edges until you can get from start to finish



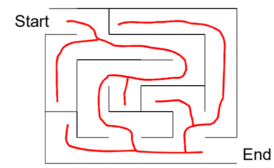
Problems with this approach

1. How can you tell when there is a path from start to finish?
   – We do not really have an algorithm yet
2. We could have *cycles*, which a "good" maze avoids
   – Want one solution and no cycles



Revised approach

- Consider edges in random order (i.e. pick an edge)
- Only delete an edge if it introduces no cycles (how? TBD)
- When done, we will have a way to get from any place to any other place (including from start to end points)



Cells and edges

- Let's number each cell
  – 36 total for 6 x 6
- An (internal) edge (x,y) is the line between cells x and y
  – 60 total for 6x6: (1,2), (2,3), …, (1,7), (2,8), …

Start	1	2	3	4	5	6	
	7	8	9	10	11	12	
	13	14	15	16	17	18	
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	End

The trick

- Partition the cells into disjoint sets
  – Two cells in same set if they are "connected"
  – Initially every cell is in its own subset
- If removing an edge would connect two different subsets:
  – then remove the edge and **union** the subsets
  – else leave the edge because removing it makes a cycle

Start	1	2	3	4	5	6	
	7	8	9	10	11	12	
	13	14	15	16	17	18	
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	End

Start	1	2	3	4	5	6	
	7	8	9	10	11	12	
	13	14	15	16	17	18	
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	End

The algorithm

- P = **disjoint sets** of connected cells
  initially each cell in its own 1-element set
- E = **set** of edges not yet processed, initially all (internal) edges
- M = **set** of edges kept in maze (initially empty)

```
while P has more than one set {
  – Pick a random edge (x,y) to remove from E
  – u = find(x)
  – v = find(y)
  – if u==v
      add (x,y) to M // same subset, do not remove edge, do not create cycle
    else
      union(u,v) // connect subsets, do not put edge in M
}
Add remaining members of E to M, then output M as the maze
```


Example

Pick edge (8,14)

Start	1	2	3	4	5	6	
	7	8	9	10	11	12	
	13	14	15	16	17	18	
	19	20	21	22	23	24	
	25	26	27	28	29	30	
	31	32	33	34	35	36	End

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

Example

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32,33,34,35,36}

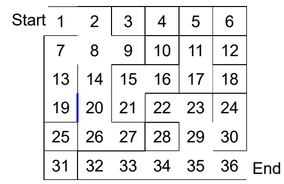
Find(8) = 7
Find(14) = 20

Union(7,20)

P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32,33,34,35,36}

Example: Add edge to M step

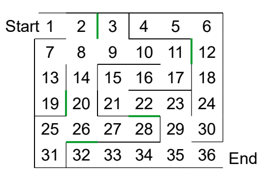
Pick edge (19,20)
Find (19) = 7
Find (20) = 7
Add (19,20) to M



P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

At the end

- Stop when P has one set (i.e. all cells connected)
- Suppose green edges are already in M and black edges were not yet picked
  - Add all black edges to M



P
{1,2,3,4,5,6,7,… 36}

Done! ☺

A data structure for the union-find ADT

- Start with an initial partition of *n* subsets
  - Often 1-element sets, e.g., {1}, {2}, {3}, …, {*n*}
- May have any number of **find** operations
- May have up to *n*-1 **union** operations in any order
  - After *n*-1 **union** operations, every **find** returns same 1 set

Teaser: the up-tree data structure

- Tree structure with:
  - No limit on branching factor
  - References from children to parent
- Start with *forest* of 1-node trees

- Possible forest after several unions:
  - Will use roots for set names