CSE373: Data Structures and Algorithms

Induction and Its Applications Part 3:

Proving Recursive Algorithms Correct with Induction

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This lecture material is based in part on materials provided by Ioana Sora at the Politechnic University of Timisoara.

Lecture Outline

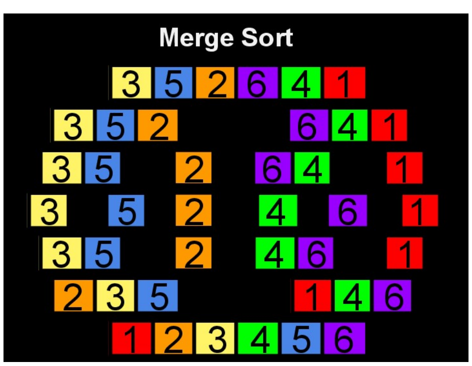
- **Proving the Correctness of Recursive Algorithms.**
- **Induction hypothesis: the recursive calls are correct.**
- **Example: Merge Sort.**
- **The General Method for Induction with Recursive Algorithms.**

Recursion is essential in applications such as creating regular fractal images.

Sierpinski-triangle bedecked umbrella image by Paul Scott.

Proof of Correctness for Recursive Algorithms

- In order to prove recursive algorithms, we have to:
  1. Prove the partial correctness (the fact that the program behaves correctly).
     - *We assume that all recursive calls with arguments that satisfy the preconditions behave as described by the specification, and use it to show that the algorithm behaves as specified.*
  2. Prove that the program terminates.
     - Any chain of recursive calls eventually ends and all loops, if any, terminate after some finite number of iterations.

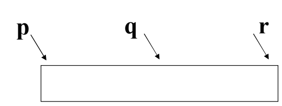


The Merge Sort algorithm is an excellent example of a recursive algorithm.

Image courtesy of cs60.net at Harvard Univ.

Example - Merge Sort

```
MERGE-SORT(A,p,r)
   if p < r
        q = (p+r)/2
        MERGE-SORT(A,p,q)
        MERGE-SORT(A,q+1,r)
        MERGE(A,p,q,r)
```



Precondition:

Array A has at least 1 element between indexes p and r ($p \leq r$).

Postcondition:

The elements between indexes p and r are sorted.

Example - Merge Sort

- MERGE-SORT calls a function MERGE(A,p,q,r) to merge the sorted subarrays of A into a single sorted one.
- The proof of MERGE (which is an iterative function) can be done separately, using loop invariants.
- We assume here that MERGE has been proved to fulfill its postconditions (can do it as a distinct exercise).

```
MERGE (A,p,q,r)

Precondition: A is an array
   and p, q, and r are
   indices into the array
   such that p ≤ q < r.
   The subarrays A[p .. q]
   and A[q+1 .. r] are sorted
Postcondition: The subarray
   A[p..r] is sorted
```


Correctness proof for Merge-Sort

- Number of elements to be sorted: n = r-p+1
- **Base Case**: n = 1
  - A contains a single element (which is trivially “sorted”).
- **Inductive Hypothesis:**
  - Assume that MergeSort correctly sorts n=1, 2, ..., **k** elements.
- **Inductive Step:**
  - Show that MergeSort correctly sorts n = **k + 1** elements.
  - First recursive call n1=q-p+1=(k+1)/2 ≤ k ⇒ subarray A[p .. q] is sorted
  - Second recursive call n2=r-q=(k+1)/2 ≤ k ⇒ subarray A[q+1 .. r] is sorted
  - A, p, q, r fulfill now the precondition of Merge
  - The postcondition of Merge guarantees that the array A[p .. r] is sorted ⇒ postcondition of MergeSort

Correctness proof for Merge-Sort

- **Termination:**
  - To argue termination, we have to find a quantity that decreases with every recursive call: the length of the part of A considered by a call to MergeSort
  - For the base case, we have a one-element array. the algorithm terminates in this case without making additional recursive calls.

Correctness proofs for recursive algorithms

```
RECURSIVE(n) is
   if (n=small_value)
      return simple_answer
   else
        RECURSIVE(n1)
        ...
        RECURSIVE(nr)
        some_code
```

$n_1, n_2, \ldots n_r$ are some values smaller than n but bigger than small_value

- **Base Case**: Prove that RECURSIVE works for n = small_value
- **Inductive Hypothesis (strong induction form):**
  - Assume that RECURSIVE works correctly for n=small_value, ..., **k**
- **Inductive Step:**
  - Show that RECURSIVE works correctly for n = **k + 1**

Lecture (Parts 1, 2, and 3) Summary

- Proving that an algorithm is totally correct means:
  1. Proving that it will ***terminate***
  2. Proving that the list of ***actions*** applied to the input (satisfying the ***precondition***) imply the output satisfies the ***postcondition.***
- How to prove ***repetitive algorithms*** correct:
  - ***Iterative*** algorithms: use ***Loop invariants***, Induction
  - ***Recursive*** algorithms: use induction using as hypothesis the recursive call

Bibliography

- Weiss, Ch. 1 section on induction.
- Goodrich and Tamassia: Induction and loop invariants; see, e.g., http://www.cs.mun.ca/~kol/courses/2711-w09/Induction.pdf)
- Erickson, J. Proof by Induction. Available at: http://jeffe.cs.illinois.edu/teaching/algorithms/notes/98-induction.pdf