CSE373: Data Structures and Algorithms

Asymptotic Analysis (Big O, \( \Omega \), and \( \Theta \))

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This lecture material represents the work of multiple instructors at the University of Washington.
Thank you to all who have contributed!

Big-O: Common Names

- \( O(1) \) constant (same as \( O(k) \) for constant \( k \))
- \( O(\log n) \) logarithmic
- \( O(n) \) linear
- \( O(n \log n) \) “\( n \log n \)”
- \( O(n^2) \) quadratic
- \( O(n^k) \) polynomial (where \( k \) is any constant)
- \( O(k^n) \) exponential (where \( k \) is any constant > 1)
- \( O(n!) \) factorial

Comparing Function Growth (e.g., for Running Times)

- For a processor capable of one million instructions per second

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \log n )</th>
<th>( n^2 )</th>
<th>( n^3 )</th>
<th>( 2^n )</th>
<th>( n! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 30 )</td>
<td>1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
</tr>
<tr>
<td>( n = 50 )</td>
<td>1 sec</td>
<td>1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
</tr>
<tr>
<td>( n = 100 )</td>
<td>1 sec</td>
<td>2 sec</td>
<td>&lt; 1 sec</td>
<td>&lt; 1 sec</td>
<td>1 sec</td>
</tr>
<tr>
<td>( n = 1,000 )</td>
<td>1 sec</td>
<td>3 sec</td>
<td>5 days</td>
<td>11 days</td>
<td>very long</td>
</tr>
<tr>
<td>( n = 10,000 )</td>
<td>1 sec</td>
<td>10 days</td>
<td>12 days</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>( n = 100,000 )</td>
<td>1 sec</td>
<td>20 days</td>
<td>31 days</td>
<td>very long</td>
<td>very long</td>
</tr>
</tbody>
</table>

Efficiency

- What does it mean for an algorithm to be efficient?
  - We care about time (and sometimes space)
- Is the following a good definition?
  - “An algorithm is efficient if, when implemented, it runs quickly on real input instances”

Gauging Performance

- Uh, why not just run the program and time it?
  - Too much variability, not reliable or portable:
    - Hardware: processor(s), memory, etc.
    - Software: OS, Java version, libraries, drivers
    - Other programs running
  - Timing does not explain relative timing among inputs (what happens when \( n \) doubles in size)
- Often want to evaluate an algorithm, not an implementation
  - Even before creating the implementation (“coding it up”)
Comparing Algorithms

When is one algorithm (not implementation) better than another?

- Various possible answers (clarity, security, …)
- But a big one is performance: for sufficiently large inputs, runs in less time (or less space)

**Large inputs** - because probably any algorithm is “plenty good” for small inputs (if \( n = 5 \), probably anything is fast)

Answer will be independent of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to “coding it up and timing it on some test cases”

Analyzing Code (“worst case”)

Basic operations take “some amount of” constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field or array index
- Etc.

(This is an approximation of reality but practical.)

<table>
<thead>
<tr>
<th>Control Flow</th>
<th>Time required</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consecutive statements</td>
<td>Sum of times of statements</td>
</tr>
<tr>
<td>Conditionals</td>
<td>Sum of times of test and slower branch</td>
</tr>
<tr>
<td>Loops</td>
<td>Number of iterations ( \times ) time of body</td>
</tr>
<tr>
<td>Calls</td>
<td>Time of called function’s body</td>
</tr>
<tr>
<td>Recursion</td>
<td>(Solve a recurrence equation)</td>
</tr>
</tbody>
</table>

Examples:

- \( 4n + 5 \) \( \in \) \( O(n) \)
- \( 0.5n \log n + 2n + 7 \) \( \in \) \( O(n \log n) \)
- \( n^2 + 2n + 3n \) \( \in \) \( O(2^n) \) EXPONENTIAL GROWTH!
- \( n \log (10n^2) + 2n \log (n) \) \( \in \) \( O(n \log n) \)

Example

Find an integer in a sorted array

```java
2 3 5 16 37 50 73 75 126
```

// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k){
    return help(arr,k,0,arr.length);
}

boolean help(int[] arr, int k, int lo, int hi){
    int mid = (hi+lo)/2;
    if(arr[mid] == k) return true;
    if(arr[mid]< k) return help(arr,k,mid+1,hi);
    else return help(arr,k,lo,mid);
}
```

Linear Search

Find an integer in a sorted array

```java
2 3 5 16 37 50 73 75 126
```

// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k){
    for(int i=0; i < arr.length; ++i)
        if(arr[i] == k)
            return true;
    return false;
}
```

Binary Search

Find an integer in a sorted array

```java
2 3 5 16 37 50 73 75 126
```

// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k, int lo, int hi){
    int mid = (hi+lo)/2;
    if(lo==hi) return false;
    if(arr[mid]==k) return true;
    if(arr[mid]< k) return help(arr,k,mid+1,hi);
    else return help(arr,k,lo,mid);
}
```
Binary Search
Best case: about 8 steps: $O(1)$
Worst case: $T(n) = 10 + T(n/2)$ where $n$ is hi-lo
- $T(n) = O(\log n)$ where $n$ is array.length
- Solve recurrence equation to know that...

```java
// requires array is sorted
// returns whether k is in array
boolean find(int[] arr, int k) {
    return help(arr, k, 0, arr.length);
}

boolean help(int[] arr, int k, int lo, int hi) {
    int mid = (hi + lo) / 2;
    if (lo == hi) return false;
    if (arr[mid] == k) return true;
    if (arr[mid] < k) return help(arr, k, mid + 1, hi);
    else return help(arr, k, lo, mid);
}
```

Best case: about 8 steps: $O(1)$
Worst case: $T(n) = 10 + T(n/2)$ = $10 + 10 + 10 + T(n/8)$
- $T(n)$ is in $O(\log n)$

Solving Recurrence Relations
1. Determine the recurrence relation. What is the base case?
   - $T(n) = 10 + T(n/2)$
   - $T(1) = 10$
2. “Expand” the original relation to find an equivalent general expression in terms of the number of expansions.
   - $T(n) = 10 + 10 + T(n/4)$
   - $T(n) = 10 + 10 + 10 + T(n/8)$
3. Find a closed-form expression by setting the number of expansions to a value which reduces the problem to a base case
   - $n(2^k) = 1$ implies $n = 2^k$ implies $k = \log n$
   - So $T(n) = 10 \log n + 8$ (get to base case and do it)
   - So $T(n)$ is in $O(\log n)$

Ignoring Constant Factors
- So binary search's runtime is in $O(\log n)$ and linear's is in $O(n)$
  - But which is faster?
- Could depend on constant factors
  - How many assignments, additions, etc. for each $n$
    - E.g. $T(n) = 5,000,000n$ vs. $T(n) = 5n^2$
    - And could depend on size of $n$
      - E.g. $T(n) = 5,000,000 + \log n$ vs. $T(n) = 10 + n$
- But there exists some $n_0$ such that for all $n > n_0$ binary search wins
- Let’s play with a couple plots to get some intuition…

Example
Let’s try to “help” linear search
Run it on a computer 100x as fast (say 2015 model vs. 1990)
Use a new compiler/language that is 3x as fast
Be a clever programmer to eliminate half the work
So doing each iteration is 600x as fast as in binary search
Note: 600x still helpful for problems without logarithmic algorithms!

Another Example: sum array
Two “obviously” linear algorithms: $T(n) = c + T(n-1)$
Iterative:
```java
int sum(int[] arr){
    int ans = 0;
    for(int i=0; i<arr.length; ++i)
        ans += arr[i];
    return ans;
}
```
Recursive:
```java
int sum(int[] arr){
    return help(arr, 0);
}
int help(int[] arr, int i){
    if(i==arr.length)
        return 0;
    return arr[i] + help(arr, i+1);
}
```
What About a Recursive Version?

```java
int sum(int[] arr) {
    return help(arr, 0, arr.length);
}
int help(int[] arr, int lo, int hi) {
    if (lo == hi) return 0;
    if (lo == hi - 1) return arr[lo];
    int mid = (hi + lo) / 2;
    return help(arr, lo, mid) + help(arr, mid, hi);
}
```

Recurrence is \( T(n) = 1 + 2T(n/2) \); \( T(1) = 1 \)

1 + 2 + 4 + 8 + \ldots \text{ for } \log n \text{ times}

\( 2^\log n - 1 \) which is proportional to \( n \) (definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": We can't do better than \( O(n) \) : we have to read whole array

Parallelism Teaser

• But suppose we could do two recursive calls at the same time
  – Like having a friend do half the work for you!

```java
int sum(int[] arr) {
    return help(arr, 0, arr.length);
}
int help(int[] arr, int lo, int hi) {
    if (lo == hi) return 0;
    if (lo == hi - 1) return arr[lo];
    int mid = (hi + lo) / 2;
    return help(arr, lo, mid) + help(arr, mid, hi);
}
```

• If you have as many "friends of friends" as needed the recurrence is now
  \( T(n) = c + T(n/2) \) – \( O(\log n) \) : same recurrence as for \( \text{find} \)

Common Recurrences

Should know how to solve recurrences but also recognize some really common ones:

- \( T(n) = c + T(n-1) \) – linear
- \( T(n) = c + 2T(n/2) \) – linear
- \( T(n) = c + T(n/2) \) – logarithmic: \( O(\log n) \)
- \( T(n) = c + 2T(n-1) \) – exponential
- \( T(n) = cn + T(n-1) \) – quadratic
- \( T(n) = cn + T(n/2) \) – linear
- \( T(n) = cn + 2T(n/2) \) – \( O(n \log n) \)

Note big-O can also use more than one variable

- Example: can sum all elements of an \( n \times m \) matrix in \( O(nm) \)

Asymptotic Notation

About to show formal definition, which amounts to saying:
1. Eliminate low-order terms
2. Eliminate coefficients

Examples:
- \( 4n + 5 \)
- \( 0.5n \log n + 2n + 7 \)
- \( n^3 + 2^n + 3n \)
- \( n \log (10n^2) \)

Big-O Relates Functions

We use \( O \) on a function \( f(n) \) (for example \( n^2 \)) to mean the set of functions with asymptotic behavior "less than or equal to" \( f(n) \)

For example, \( (3n^2 + 17) \) is in \( O(n^2) \)

Confusingly, some people also say/write:
- \( (3n^2 + 17) \) is \( O(n^2) \)
- \( (3n^2 + 17) \) = \( O(n^2) \)

But we should never say \( O(n^2) = (3n^2 + 17) \)

Big-O, Formally

Definition:
\( f(n) \) is in \( O(g(n)) \) if there exist constants \( c \) and \( n_0 \) such that \( f(n) \leq c g(n) \) for all \( n \geq n_0 \)

- To show \( f(n) \) is in \( O(g(n)) \), pick a \( c \) large enough to "cover the constant factors" and \( n_0 \) large enough to "cover the lower-order terms"
- Example: Let \( f(n) = 3n^2 + 17 \) and \( g(n) = n^3 \)
  - \( c=5 \) and \( n_0=10 \) will work.
- This is "less than or equal to"
  - So \( 3n^2 + 17 \) is also in \( O(n^3) \) and in \( O(2^n) \) etc.
More Examples, Using Formal Definition

• Let \( f(n) = 1000n \) and \( g(n) = n^2 \)
  – A valid proof is to find valid \( c \) and \( n_0 \)
  – The “cross-over point” is \( n=1000 \)
  – So we can choose \( n_0=1000 \) and \( c=1 \)

Many other possible choices, e.g., larger \( n_0 \) and/or \( c \)

What’s with the \( c \)?

• The constant multiplier \( c \) is what allows functions that differ only in their largest coefficient to have the same asymptotic complexity
• Example: \( f(n) = 7n+5 \) and \( g(n) = n \)
  – For any choice of \( n_0 \), need a \( c \geq 7 \) (or more) to show \( f(n) \) is in \( O(g(n)) \)

What You Can Drop

• Eliminate coefficients because we don’t have units anyway
  – \( 3n^2 \) versus \( 5n^2 \) doesn’t mean anything when we have not specified the cost of constant-time operations (can re-scale)

• Eliminate low-order terms because they have vanishingly small impact as \( n \) grows

• Do NOT ignore constants that are not multipliers
  – \( n^2 \) is not in \( O(n^2) \)
  – \( 3^3 \) is not in \( O(2^n) \)

(This all follows from the formal definition)

Aesthetics of a Big-O Demonstrations

• Sometimes, \( f(n) \) is clearly “dominated” by \( g(n) \).
• That happens when \( f \) is in \( O(g) \), but \( g \) is not in \( O(f) \).
• For example \( 2n \) is in \( O(n^2) \) but \( n^2 \) is not in \( O(2n) \).
• Then to show \( f(n) \) is in \( O(g(n)) \), it is good form to use \( c = 1 \) and the smallest \( n_0 \) that works with it, or the smallest integer value of \( n_0 \) that works with it.
• We show \( 2n \) is in \( O(n^2) \) by taking \( c=1 \) and \( n_0 = 2^{1/2} \) or \( n_0 = 2 \).

Big-O: Common Names (Again)

\begin{align*}
\text{Big-O: Common Names (Again)}
O(1) & \quad \text{constant} \\
O(\log n) & \quad \text{logarithmic} \\
O(n) & \quad \text{linear} \\
O(n \log n) & \quad "n \log n" \\
O(n^2) & \quad \text{quadratic} \\
O(n^3) & \quad \text{cubic} \\
O(n^k) & \quad \text{polynomial (where } k \text{ is any constant)} \\
O(k^n) & \quad \text{exponential (where } k \text{ is any constant } > 1 \text{)}
\end{align*}
More Asymptotic Notation

• Upper bound: $O(g(n))$ is the set of all functions asymptotically less than or equal to $f(g)$
  - $f(n)$ is in $O(g(n))$ if there exist constants $c$ and $n_0$ such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$

• Lower bound: $\Omega(g(n))$ is the set of all functions asymptotically greater than or equal to $g(n)$
  - $f(n)$ is in $\Omega(g(n))$ if there exist constants $c$ and $n_0$ such that $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$

• Tight bound: $\Theta(g(n))$ is the set of all functions asymptotically equal to $g(n)$
  - Intersection of $O(g(n))$ and $\Omega(g(n))$ (use different $c$ values)

Correct Terms, in Theory

A common error is to say $O(f(n))$ when you mean $\Theta(f(n))$
- Since a linear algorithm is also $O(n^2)$, it’s tempting to say “this algorithm is exactly $O(n^2)$”
- That doesn’t mean anything, say it is $O(n)$
- That means that it is not, for example $O(\log n)$

Less common notation:
- “little-oh”: intersection of “big-Oh” and not “big-Theta”
  - For all $c$, there exists an $n_0$ such that $f(n) \leq c \cdot g(n)$
  - Example: array sum is $o(n^2)$ but not $o(n)$
- “strictly greater than”
  - “little-omega”: intersection of “big-Omega” and not “big-Theta”
  - For all $c$, there exists an $n_0$ such that $c \cdot g(n) \leq f(n)$
  - Example: array sum is $\omega(\log n)$ but not $\omega(n)$
- “strictly less than”

What We Are Analyzing

• The most common thing to do is give an $O$ or $\Theta$ bound to the worst-case running time of an algorithm

• Example: binary-search algorithm
  - Common: $(\log n)$ running-time in the worst-case
  - Less common: $\Theta(1)$ in the best-case (item is in the middle)
  - Less common (but very good to know): the find-in-sorted-array problem is $\Theta(\log n)$ in the worst-case
  - No algorithm can do better
  - A problem cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean there exists an algorithm

Other Things to Analyze

• Space instead of time
  - Remember we can often use space to gain time.

• Average case
  - Sometimes only if you assume something about the probability distribution of inputs
  - Sometimes uses randomization in the algorithm
  - Will see an example with sorting
  - Sometimes an amortized guarantee
  - Average time over any sequence of operations
  - Will discuss in a later lecture

Summary

Analysis can be about:
• The problem or the algorithm (usually algorithm)
• Time or space (usually time)
  - Or power or dollars or …
• Best-, worst-, or average-case (usually worst)
• Upper-, lower-, or tight-bound (usually upper or tight)

Usually Asymptotic Analysis is Valuable

• Asymptotic complexity focuses on behavior for large $n$ and is independent of any computer / coding trick.
• But you can “abuse” it to be misled about trade-offs.
  - Example: $n^{10}$ vs. $\log n$
    - Asymptotically $n^{10}$ grows more quickly.
    - But the “cross-over” point is around $5 \times 10^{17}$
    - So if you have input size less than $2^{50}$, prefer $n^{10}$
  - For small $n$, an algorithm with worse asymptotic complexity might be faster.
    - Here the constant factors can matter, if you care about performance for small $n$. 
Timing vs. Big-O Summary

• Big-O notation is an essential part of computer science’s mathematical foundation
  – Examine the algorithm itself, not the implementation.
  – Reason about (even prove) performance as a function of $n$.

• Timing also has its place
  – Compare implementations.
  – Focus on data sets you care about (versus worst case).
  – Determine what the constant factors "really are".