CSE373: Data Structure & Algorithms
Lecture 22: More Sorting

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Admin

- Homework 5 partner selection due TODAY!
  - Catalyst link posted on the webpage

- Homework 5 due next Wednesday at 11pm!
The comparison sorting problem

Assume we have \( n \) comparable elements in an array and we want to rearrange them to be in increasing order.

Input:
- An array \( A \) of data records
- A key value in each data record
- A comparison function (consistent and total)

Effect:
- Reorganize the elements of \( A \) such that for any \( i \) and \( j \), if \( i < j \) then \( A[i] \leq A[j] \)
- (Also, \( A \) must have exactly the same data it started with)
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort.
**Sorting: The Big Picture**

Surprising amount of neat stuff to say about sorting:

- **Simple algorithms:** $O(n^2)$
  - Insertion sort
  - Selection sort
  - Shell sort
  - ...

- **Fancier algorithms:** $O(n \log n)$
  - Heap sort
  - Merge sort
  - Quick sort
  - ...

- **Comparison lower bound:** $\Omega(n \log n)$

- **Specialized algorithms:** $O(n)$
  - Bucket sort
  - Radix sort

- **Handling huge data sets**
  - External sorting
Divide and conquer

Very important technique in algorithm design

1. Divide problem into smaller parts

2. Independently solve the simpler parts
   - Think recursion
   - Or potential parallelism

3. Combine solution of parts to produce overall solution
Divide-and-Conquer Sorting

Two great sorting methods are fundamentally divide-and-conquer

1. **Merge sort:** Sort the left half of the elements (recursively)  
   Sort the right half of the elements (recursively)  
   Merge the two sorted halves into a sorted whole

2. **Quick sort:** Pick a “pivot” element  
   Divide elements into less-than pivot and greater-than pivot  
   Sort the two divisions (recursively on each)  
   Answer is sorted-less-than then pivot then sorted-greater-than
Merge sort

- To sort array from position \(lo\) to position \(hi\):
  - If range is 1 element long, it is already sorted! (Base case)
  - Else:
    - Sort from \(lo\) to \((hi+lo)/2\)
    - Sort from \((hi+lo)/2\) to \(hi\)
    - Merge the two halves together

- Merging takes two sorted parts and sorts everything
  - \(O(n)\) but requires auxiliary space…
Some details: saving a little time

• What if the final steps of our merge looked like this:

![Array Diagram]

Main array

Auxiliary array

• Wasteful to copy to the auxiliary array just to copy back…
Some details: saving a little time

- If left-side finishes first, just stop the merge and copy back:

- If right-side finishes first, copy dregs into right then copy back
Some details: Saving Space and Copying

Simplest / Worst:
   Use a new auxiliary array of size \((hi-lo)\) for every merge

Better:
   Use a new auxiliary array of size \(n\) for every merging stage

Better:
   Reuse same auxiliary array of size \(n\) for every merging stage

Best (but a little tricky):
   Don’t copy back – at 2\(^{nd}\), 4\(^{th}\), 6\(^{th}\), … merging stages, use the original array as the auxiliary array and vice-versa
      – Need one copy at end if number of stages is odd
Swapping Original / Auxiliary Array ("best")

- First recurse down to lists of size 1
- As we return from the recursion, swap between arrays

(Arguably easier to code up without recursion at all)
Linked lists and big data

We defined sorting over an array, but sometimes you want to sort linked lists

One approach:
- Convert to array: $O(n)$
- Sort: $O(n \log n)$
- Convert back to list: $O(n)$

Or merge sort works very nicely on linked lists directly
- Heapsort and quicksort do not
- Insertion sort and selection sort do but they’re slower

Merge sort is also the sort of choice for external sorting
- Linear merges minimize disk accesses
- And can leverage multiple disks to get streaming accesses
Quick sort

• A divide-and-conquer algorithm
  – Recursively chop into two pieces
  – Instead of doing all the work as we merge together, we will do all the work as we recursively split into halves
  – Unlike merge sort, does not need auxiliary space

• $O(n \log n)$ on average 😊, but $O(n^2)$ worst-case 😞

• Faster than merge sort in practice?
  – Often believed so
  – Does fewer copies and more comparisons, so it depends on the relative cost of these two operations!
Quicksort Overview

1. Pick a pivot element

2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot

3. Recursively sort A and C

4. The answer is, “as simple as A, B, C”
Think in Terms of Sets

Select pivot value

Partition S

Quicksort(S₁) and Quicksort(S₂)

Presto! S is sorted

[Weiss]
Example, Showing Recursion

Divide

Divide

Divide

1 Element

Conquer

Conquer

Conquer

Conquer
Details

Have not yet explained:

- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about equal in size

- How to implement partitioning
  - In linear time
  - In place
Pivots

• Best pivot?
  – Median
  – Halve each time

• Worst pivot?
  – Greatest/least element
  – Problem of size n - 1
  – $O(n^2)$
Potential pivot rules

While sorting \texttt{arr} from \texttt{lo} to \texttt{hi-1} …

• Pick \texttt{arr[lo]} or \texttt{arr[hi-1]}
  – Fast, but worst-case occurs with mostly sorted input

• Pick random element in the range
  – Does as well as any technique, but (pseudo)random number generation can be slow
  – Still probably the most elegant approach

• Median of 3, e.g., \texttt{arr[lo]}, \texttt{arr[hi-1]}, \texttt{arr[(hi+lo)/2]}
  – Common heuristic that tends to work well
Partitioning

• Conceptually simple, but hardest part to code up correctly
  – After picking pivot, need to partition in linear time in place

• One approach (there are slightly fancier ones):
  1. Swap pivot with \( \text{arr}[lo] \)
  2. Use two fingers \( i \) and \( j \), starting at \( lo+1 \) and \( hi-1 \)
  3. \textbf{while} \( (i < j) \)
     \hspace{1em} if \( (\text{arr}[j] > \text{pivot}) \) \( j-- \)
     \hspace{1em} else if \( (\text{arr}[i] < \text{pivot}) \) \( i++ \)
     \hspace{1em} else swap \( \text{arr}[i] \) with \( \text{arr}[j] \)
  4. Swap pivot with \( \text{arr}[i] \) *

*skip step 4 if pivot ends up being least element
Example

• Step one: pick pivot as median of 3
  – $lo = 0$, $hi = 10$

  \[
  \begin{array}{cccccccccc}
  0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  8 & 1 & 4 & 9 & \text{**0**} & 3 & 5 & 2 & 7 & 6 \\
  \end{array}
  \]

• Step two: move pivot to the $lo$ position

  \[
  \begin{array}{cccccccccc}
  0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
  6 & 1 & 4 & 9 & 0 & 3 & 5 & 2 & 7 & 8 \\
  \end{array}
  \]
Example

Now partition in place

Move fingers

Swap

Move fingers

Move pivot

Often have more than one swap during partition – this is a short example
Quick sort visualization

Analysis

- Best-case: Pivot is always the median
  \[ T(0) = T(1) = 1 \]
  \[ T(n) = 2T(n/2) + n \] -- linear-time partition
  Same recurrence as merge sort: \( O(n \log n) \)

- Worst-case: Pivot is always smallest or largest element
  \[ T(0) = T(1) = 1 \]
  \[ T(n) = T(n-1) + n \]
  Basically same recurrence as selection sort: \( O(n^2) \)

- Average-case (e.g., with random pivot)
  \( O(n \log n) \), not responsible for proof (in text)
**Cutoffs**

- For small $n$, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for large $n$

- Common engineering technique: switch algorithm below a cutoff
  - Reasonable rule of thumb: use insertion sort for $n < 10$

- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - Switch to sequential algorithm
  - None of this affects asymptotic complexity
Cutoff pseudocode

```c
void quicksort(int[] arr, int lo, int hi) {
    if(hi - lo < CUTOFF)
        insertionSort(arr,lo,hi);
    else
        ...
}
```

Notice how this cuts out the vast majority of the recursive calls
- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree
How Fast Can We Sort?

- Heapsort & mergesort have $O(n \log n)$ worst-case running time
- Quicksort has $O(n \log n)$ average-case running time
- These bounds are all tight, actually $\Theta(n \log n)$
- Comparison sorting in general is $\Omega(n \log n)$
  - An amazing computer-science result: proves all the clever programming in the world cannot comparison-sort in linear time
The Big Picture

Surprising amount of juicy computer science: 2-3 lectures…

- Simple algorithms: $O(n^2)$
  - Insertion sort
  - Selection sort
  - Shell sort
  - ...

- Fancier algorithms: $O(n \log n)$
  - Heap sort
  - Merge sort
  - Quick sort (avg)
  - …

- Comparison lower bound: $\Omega(n \log n)$

- Specialized algorithms: $O(n)$
  - Bucket sort
  - Radix sort

- Handling huge data sets
  - External sorting

How???
- Change the model – assume more than “compare(a,b)”
Bucket Sort (a.k.a. BinSort)

- If all values to be sorted are known to be integers between 1 and $K$ (or any small range):
  - Create an array of size $K$
  - Put each element in its proper bucket (a.k.a. bin)
  - If data is only integers, no need to store more than a count of how times that bucket has been used
- Output result via linear pass through array of buckets

<table>
<thead>
<tr>
<th>count array</th>
<th>1</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

- Example:
  - $K=5$
  - input (5,1,3,4,3,2,1,1,5,4,5)
  - output: 1,1,1,2,3,3,4,4,5,5,5
Visualization

Analyzing Bucket Sort

• Overall: $O(n+K)$
  – Linear in $n$, but also linear in $K$
  – $\Omega(n \log n)$ lower bound does not apply because this is not a comparison sort

• Good when $K$ is smaller (or not much larger) than $n$
  – We don’t spend time doing comparisons of duplicates

• Bad when $K$ is much larger than $n$
  – Wasted space; wasted time during linear $O(K)$ pass

• For data in addition to integer keys, use list at each bucket
Bucket Sort with Data

• Most real lists aren’t just keys; we have data
• Each bucket is a list (say, linked list)
• To add to a bucket, insert in $O(1)$ (at beginning, or keep pointer to last element)

<table>
<thead>
<tr>
<th>count array</th>
<th>Rocky V</th>
<th>Harry Potter</th>
<th>Casablanca</th>
<th>Star Wars</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
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<td>5</td>
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</tbody>
</table>

• Example: Movie ratings; scale 1-5; 1=bad, 5=excellent
  Input=
  5: Casablanca
  3: Harry Potter movies
  5: Star Wars Original Trilogy
  1: Rocky V

• Result: 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars
• Easy to keep ‘stable’; Casablanca still before Star Wars
Radix sort

- Radix = “the base of a number system”
  - Examples will use 10 because we are used to that
  - In implementations use larger numbers
    - For example, for ASCII strings, might use 128

- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with least significant digit
    - Keeping sort stable
  - Do one pass per digit
  - Invariant: After $k$ passes (digits), the last $k$ digits are sorted

- Aside: Origins go back to the 1890 U.S. census
Example

Radix = 10

Input: 478
537
9
721
3
38
143
67

Order now: 721
3
143
537
67
478
38
9

First pass:
bucket sort by ones digit
Example

Radix = 10

Order was: 721 3 143 537 67 478 38 9

Second pass:
stable bucket sort by tens digit

Order now: 721 3 537 67 478 9

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### Example

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>143</td>
<td></td>
<td>67</td>
<td>478</td>
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<td>9</td>
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</tr>
</tbody>
</table>

Radix = 10

Order was:

3
9
721
537
38
67

Third pass:

*stable* bucket sort by 100s digit

Order now:

3
9
38
67
143
478
537
721
Visualization

Analysis

Input size: $n$
Number of buckets = Radix: $B$
Number of passes = “Digits”: $P$

Work per pass is 1 bucket sort: $O(B+n)$

Total work is $O(P(B+n))$

Compared to comparison sorts, sometimes a win, but often not

- Example: Strings of English letters up to length 15
  - Run-time proportional to: $15*(52 + n)$
  - This is less than $n \log n$ only if $n > 33,000$
  - Of course, cross-over point depends on constant factors of the implementations
    - And radix sort can have poor locality properties
Sorting massive data

• Need sorting algorithms that minimize disk/tape access time:
  – Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
  – Merge sort scans linearly through arrays, leading to (relatively) efficient sequential disk access

• Merge sort is the basis of massive sorting

• Merge sort can leverage multiple disks
External Merge Sort

- Sort 900 MB using 100 MB RAM
  - Read 100 MB of data into memory
  - Sort using conventional method (e.g. quicksort)
  - Write sorted 100MB to temp file
  - Repeat until all data in sorted chunks ($900/100 = 9$ total)
- Read first 10 MB of each sorted chunk, merge into remaining 10MB
  - writing and reading as necessary
  - Single merge pass instead of $\log n$
  - Additional pass helpful if data much larger than memory
- Parallelism and better hardware can improve performance
- Distribution sorts (similar to bucket sort) are also used
Last Slide on Sorting

• Simple $O(n^2)$ sorts can be fastest for small $n$
  – Selection sort, Insertion sort (latter linear for mostly-sorted)
  – Good for “below a cut-off” to help divide-and-conquer sorts
• $O(n \log n)$ sorts
  – Heap sort, in-place but not stable nor parallelizable
  – Merge sort, not in place but stable and works as external sort
  – Quick sort, in place but not stable and $O(n^2)$ in worst-case
    • Often fastest, but depends on costs of comparisons/copies
• $\Omega(n \log n)$ is worst-case and average lower-bound for sorting by comparisons
• Non-comparison sorts
  – Bucket sort good for small number of possible key values
  – Radix sort uses fewer buckets and more phases
• Best way to sort? It depends!