CSE373: Data Structures and Algorithms

Lecture 4: Asymptotic Analysis

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Previously, on CSE 373

- We want to analyze algorithms for efficiency (in time and space)
- And do so generally and rigorously
  - not timing an implementation
- We will primarily consider worst-case running time
- Example: find an integer in a sorted array
  - Linear search: $O(n)$
  - Binary search: $O(\texttt{log}\ n)$
  - Had to solve a recurrence relation to see this

Another example: sum array

Two "obviously" linear algorithms: $T(n) = O(1) + T(n\text{-}1)$

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}
```

Recursive:

– Recurrence is $k + k + \ldots + k$ for n times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
     return 0;
  return arr[i] + help(arr,i+1);
}
```

What about a binary version?

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$

- 1 + 2 + 4 + 8 + …   for **`log`** *n* times
- $2^{(\log n)} - 1$ which is proportional to *n* (definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than *O(n)* – have to read whole array

Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
  - *Like having a friend do half the work for you!*

```
int sum(int[]arr){
   return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed the recurrence is now $T(n) = O(1) + 1\,T(n/2)$
  - $O($`log` $n)$: same recurrence as for **`find`**

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic (see previous lecture)
$T(n) = O(n) + 2T(n/2)$	O(n `log` n)

Note big-Oh can also use more than one variable

- Example: can sum all elements of an *n*-by-*m* matrix in $O(nm)$

Asymptotic notation

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$
- $0.5n \log n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \log (10n^2)$

Big-Oh relates functions

We use O on a function f(n) (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** f(n)

So ($3n^2$+17) **is in** $O(n^2)$

- $3n^2$+17 and n^2 have the same asymptotic behavior

Confusingly, we also say/write:

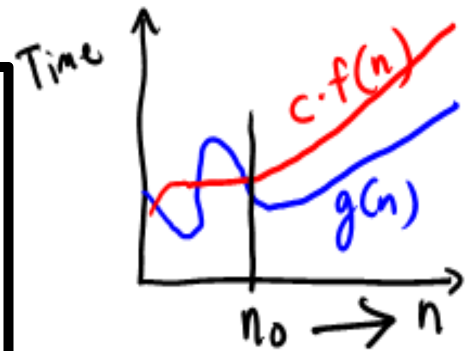
- ($3n^2$+17) **is** $O(n^2)$
- ($3n^2$+17) = $O(n^2)$

But we would never say $O(n^2)$ = ($3n^2$+17)

Big-O, formally

Definition:

g(n) is in O(f(n)) if there exist constants c and n_0 such that g(n) $\leq$ c f(n) for all $n \geq n_0$



- To show g(n) is in O(f(n)), pick a c large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"
  - Example: Let g(n) = $3n^2+17$ and f(n) = n^2

    c=5 and n_0=10 is more than good enough

- This is "less than or equal to"
  - So $3n^2+17$ is also $O(n^5)$ and $O(2^n)$ etc.

More examples, using formal definition

- Let $g(n) = 1000n$ and $f(n) = n^2$
  - A valid proof is to find valid c and n_0
  - The "cross-over point" is n=1000
  - So we can choose n_0=1000 and c=1
    - Many other possible choices, e.g., larger n_0 and/or c

> Definition:
>
> $g(n)$ is in $O(\ f(n)\ )$ if there exist constants
> c and n_0 such that $g(n) \leq c\ f(n)$ for all $n \geq n_0$

More examples, using formal definition

- Let $g(n) = n^4$ and $f(n) = 2^n$
  - A valid proof is to find valid c and n_0
  - We can choose n_0=20 and c=1

Definition:

g(n) is in O(f(n)) if there exist constants c and n_0 such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$

What's with the c

- The constant multiplier c is what allows functions that differ only in their largest coefficient to have the same asymptotic complexity
- Example: $g(n) = 7n+5$ and $f(n) = n$
  - For any choice of n_0, need a $c > 7$ (or more) to show $g(n)$ is in $O(f(n))$

Definition:

$g(n)$ is in $O(f(n))$ if there exist constants c and n_0 such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$

What you can drop

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we have not specified the cost of constant-time operations (can re-scale)

- Eliminate low-order terms because they have vanishingly small impact as n grows

- Do NOT ignore constants that are not multipliers
  - n^3 is not $O(n^2)$
  - 3^n is not $O(2^n)$

(This all follows from the formal definition)

Big-O: Common Names (Again)

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\texttt{log}\ n)$	logarithmic (probing)
$O(n)$	linear (single-pass)
$O(n\ \texttt{log}\ n)$	"$n\ \texttt{log}\ n$" (mergesort)
$O(n^2)$	quadratic (nested loops)
$O(n^3)$	cubic (more nested loops)
$O(n^k)$	polynomial (where is k is any constant)
$O(k^n)$	exponential (where k is any constant > 1)

Big-O running times

- For a processor capable of one million instructions per second

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

More Asymptotic Notation

- Upper bound: $O(\ f(n)\ )$ is the set of all functions asymptotically less than or equal to $f(n)$
  - $g(n)$ is in $O(\ f(n)\ )$ if there exist constants c and n_0 such that $g(n) \leq c\ f(n)$ for all $n \geq n_0$
- Lower bound: $\Omega(\ f(n)\ )$ is the set of all functions asymptotically greater than or equal to $f(n)$
  - $g(n)$ is in $\Omega(\ f(n)\ )$ if there exist constants c and n_0 such that $g(n) \geq c\ f(n)$ for all $n \geq n_0$
- Tight bound: $\theta(\ f(n)\ )$ is the set of all functions asymptotically equal to $f(n)$
  - Intersection of $O(\ f(n)\ )$ and $\Omega(\ f(n)\ )$ (use *different* c values)

Correct terms, in theory

A common error is to say $O(f(n))$ when you mean $\theta(f(n))$

- Since a linear algorithm is also $O(n^5)$, it's tempting to say "this algorithm is exactly $O(n)$"
- That doesn't mean anything, say it is $\theta(n)$
- That means that it is not, for example $O(\log n)$

Less common notation:

- "little-oh": intersection of "big-Oh" and *not* "big-Theta"
  - *For all c*, there exists an n_0 such that… $\leq$
  - Example: array sum is $o(n^2)$ but not $o(n)$
- "little-omega": intersection of "big-Omega" and *not* "big-Theta"
  - *For all c*, there exists an n_0 such that… $\geq$
  - Example: array sum is $\omega(\log n)$ but not $\omega(n)$

What we are analyzing

- The most common thing to do is give an *O* or θ bound to the worst-case running time of an algorithm

- Example: binary-search algorithm
  - Common: θ(`log` *n*) running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common: Algorithm is Ω(`log log` *n*) in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is Ω(`log` *n*) in the worst-case
    - *No* algorithm can do better
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time

- Average case
  - Sometimes only if you assume something about the *probability distribution* of inputs
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting
  - Sometimes an *amortized guarantee*
    - Average time over any sequence of operations
    - Will discuss in a later lecture

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)

Usually asymptotic is valuable

- Asymptotic complexity focuses on behavior for large *n* and is independent of any computer / coding trick

- But you can "abuse" it to be misled about trade-offs

- Example: $n^{1/10}$ vs. `log` *n*
  - Asymptotically $n^{1/10}$ grows more quickly
  - But the "cross-over" point is around $5 * 10^{17}$
  - So if you have input size less than 2^{58}, prefer $n^{1/10}$

- For *small n*, an algorithm with worse asymptotic complexity might be faster
  - Here the constant factors can matter, if you care about performance for small *n*

Timing vs. Big-Oh Summary

- Big-oh is an essential part of computer science's mathematical foundation
  - Examine the algorithm itself, not the implementation
  - Reason about (even prove) performance as a function of *n*

- Timing also has its place
  - Compare implementations
  - Focus on data sets you care about (versus worst case)
  - Determine what the constant factors "really are"