CSE373: Data Structures and Algorithms

Lecture 3: Asymptotic Analysis

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Efficiency

- What does it mean for an algorithm to be *efficient*?
  - We primarily care about *time* (and sometimes *space*)
- Is the following a good definition?
  - "An algorithm is efficient if, when implemented, it runs quickly on real input instances"
  - Where and how well is it implemented?
  - What constitutes "real input?"
  - How does the algorithm *scale* as input size changes?

Gauging efficiency (performance)

- Why not just run the program and time it?
  - Too much *variability*, not reliable or *portable*:
    - Hardware: processor(s), memory, etc.
    - OS, Java version, libraries, drivers
    - Other programs running
    - Implementation dependent
  - Choice of input
    - Testing (inexhaustive) may *miss* worst-case input
    - Timing does not *explain* relative timing among inputs (what happens when *n* doubles in size)

- Often want to evaluate an *algorithm*, not an implementation
  - Even *before* creating the implementation ("coding it up")

Comparing algorithms

When is one *algorithm* (not *implementation*) better than another?

- Various possible answers (clarity, security, …)
- But a big one is *performance*: for sufficiently large inputs, runs in less time (our focus) or less space

Large inputs because probably any algorithm is "plenty good" for small inputs (if n is 10, probably anything is fast)

Answer will be *independent* of CPU speed, programming language, coding tricks, etc.

Answer is general and rigorous, complementary to "coding it up and timing it on some test cases"

Why worst-case running times?

- Has proven reasonable in practice
- Difficult to find a satisfactory alternative
  - What about average case?
  - Difficult to express full range of input
  - Could we use randomly-generated input?
  - May learn more about generator than algorithm

Analyzing code ("worst case")

Basic operations take "some amount of" constant time

- Arithmetic (fixed-width)
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of times
Conditionals	Time of test plus slower branch
Loops	Sum of iterations
Calls	Time of call's body
Recursion	Solve *recurrence equation*

Example

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
    ???
}
```

Linear search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6ish steps = $O(1)$

Worst case: 6ish*(arr.length)

= O(arr.length)

Binary search

2	3	5	16	37	50	73	75	126

Find an integer in a *sorted* array

- Can also be done non-recursively but “doesn’t matter” here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; // i.e., lo+(hi-lo)/2
   if(lo==hi)          return false;
   if(arr[mid]==k)     return true;
   if(arr[mid]< k)     return help(arr,k,mid+1,hi);
   else                return help(arr,k,lo,mid);
}
```

Binary search

Best case: 8ish steps = $O(1)$

Worst case: $T(n) = 10\text{ish} + T(n/2)$ where n is **`hi-lo`**

- $O($**`log`** $n)$ where n is **`array.length`**
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)       return false;
   if(arr[mid]==k)  return true;
   if(arr[mid]< k)  return help(arr,k,mid+1,hi);
   else             return help(arr,k,lo,mid);
}
```

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   - $T(n) = 10 + T(n/2)$ $\quad T(1) = 8$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   - $T(n) = 10 + 10 + T(n/4)$
     $= 10 + 10 + 10 + T(n/8)$
     $= \ldots$
     $= 10k + T(n/(2^k))$
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   - So $T(n) = 10 \log_2 n + 8$ (get to base case and do it)
   - So $T(n)$ is $O(\log n)$

Ignoring constant factors

- So binary search is $O(\texttt{log}\ n)$ and linear is $O(n)$
  - But which is faster?

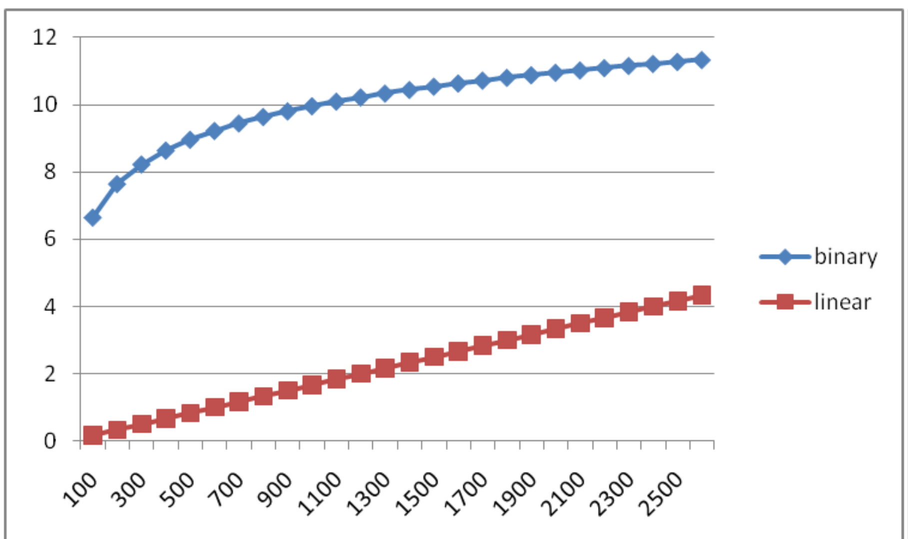
- Could depend on constant factors
  - How *many* assignments, additions, etc. for each n
    - *E.g.* $T(n) = 5{,}000{,}000n$ vs. $T(n) = 5n^2$
  - And could depend on overhead unrelated to n
    - *E.g.* $T(n) = 5{,}000{,}000 + \log n$ vs. $T(n) = 10 + n$

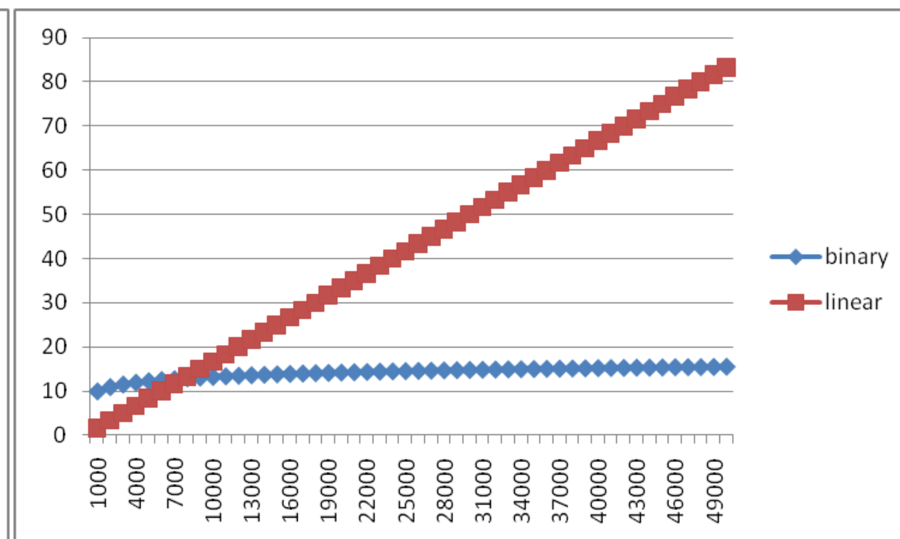
- But there exists some n_0 such that for all $n > n_0$ binary search wins

- Let's play with a couple plots to get some intuition…

Example

- Let's try to "help" linear search
  - Run it on a computer 100x as fast (say 2010 model vs. 1990)
  - Use a new compiler/language that is 3x as fast
  - Be a clever programmer to eliminate half the work
  - So doing each iteration is 600x as fast as in binary search





Another example: sum array

Two "obviously" linear algorithms: $T(n) = O(1) + T(n\text{-}1)$

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}
```

Recursive:

– Recurrence is $k + k + \ldots + k$ for n times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

What about a binary version?

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$

- $1 + 2 + 4 + 8 + \ldots$ for `log` n times
- $2^{(\log n)} - 1$ which is proportional to n (definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than *O(n)* – have to read whole array

Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
  - *Like having a friend do half the work for you!*

```
int sum(int[]arr){
   return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed the recurrence is now $T(n) = O(1) + 1\,T(n/2)$
  - $O($**`log`** $n)$: same recurrence as for **`find`**

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$T(n) = O(1) + T(n\text{-}1)$	linear
$T(n) = O(1) + 2T(n/2)$	linear
$T(n) = O(1) + T(n/2)$	logarithmic
$T(n) = O(1) + 2T(n\text{-}1)$	exponential
$T(n) = O(n) + T(n\text{-}1)$	quadratic (see previous lecture)
$T(n) = O(n) + 2T(n/2)$	O(n **`log`** n)

Note big-Oh can also use more than one variable

- Example: can sum all elements of an n-by-m matrix in $O(nm)$

Asymptotic notation

About to show formal definition, which amounts to saying:

1. Eliminate low-order terms
2. Eliminate coefficients

Examples:

- $4n + 5$
- $0.5n \texttt{log}\ n + 2n + 7$
- $n^3 + 2^n + 3n$
- $n \texttt{log}\ (10n^2)$

Big-Oh relates functions

We use O on a function $f(n)$ (for example n^2) to mean *the set of functions with asymptotic behavior* ***less than or equal to*** $f(n)$

So $(3n^2+17)$ **is in** $O(n^2)$

- $3n^2+17$ and n^2 have the same asymptotic behavior

Confusingly, we also say/write:

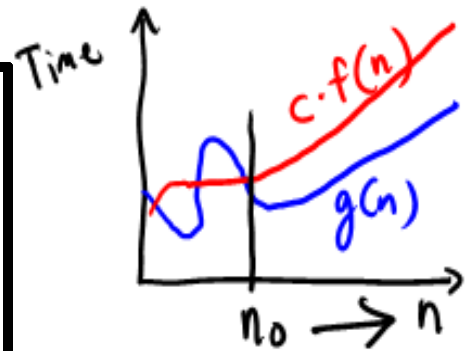
- $(3n^2+17)$ **is** $O(n^2)$
- $(3n^2+17) = O(n^2)$

But we would never say $O(n^2) = (3n^2+17)$

Big-O, formally

Definition:

g(n) is in O(f(n)) if there exist constants c and n_0 such that g(n) $\le$ c f(n) for all $n \ge n_0$



- To show g(n) is in O(f(n)), pick a c large enough to "cover the constant factors" and n_0 large enough to "cover the lower-order terms"
  - Example: Let g(n) = $3n^2+17$ and f(n) = n^2

    c=5 and n_0=10 is more than good enough

- This is "less than or equal to"
  - So $3n^2+17$ is also $O(n^5)$ and $O(2^n)$ etc.

More examples, using formal definition

- Let g(n) = 1000n and f(n) = n^2
  - A valid proof is to find valid c and n_0
  - The "cross-over point" is n=1000
  - So we can choose n_0=1000 and c=1
    - Many other possible choices, e.g., larger n_0 and/or c

Definition:

g(n) is in O(f(n)) if there exist constants c and n_0 such that g(n) $\leq$ c f(n) for all $n \geq n_0$

More examples, using formal definition

- Let $g(n) = n^4$ and $f(n) = 2^n$
  - A valid proof is to find valid c and n_0
  - We can choose n_0=20 and c=1

Definition:

g(n) is in O(f(n)) if there exist constants c and n_0 such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$

What's with the c

- The constant multiplier c is what allows functions that differ only in their largest coefficient to have the same asymptotic complexity
- Example: $g(n) = 7n+5$ and $f(n) = n$
  - For any choice of n_0, need a $c > 7$ (or more) to show $g(n)$ is in $O(f(n))$

Definition:

$g(n)$ is in $O(f(n))$ if there exist constants c and n_0 such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$

What you can drop

- Eliminate coefficients because we don't have units anyway
  - $3n^2$ versus $5n^2$ doesn't mean anything when we have not specified the cost of constant-time operations (can re-scale)

- Eliminate low-order terms because they have vanishingly small impact as n grows

- Do NOT ignore constants that are not multipliers
  - n^3 is not $O(n^2)$
  - 3^n is not $O(2^n)$

(This all follows from the formal definition)

Big-O: Common Names (Again)

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\log n)$	logarithmic
$O(n)$	linear
$O(n \log n)$	"n log n"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant)
$O(k^n)$	exponential (where k is any constant > 1)

Big-O running times

- For a processor capable of one million instructions per second

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

More Asymptotic Notation

- Upper bound: $O(\ f(n)\ )$ is the set of all functions asymptotically less than or equal to $f(n)$
  - $g(n)$ is in $O(\ f(n)\ )$ if there exist constants c and n_0 such that $g(n) \leq c\, f(n)$ for all $n \geq n_0$
- Lower bound: $\Omega(\ f(n)\ )$ is the set of all functions asymptotically greater than or equal to $f(n)$
  - $g(n)$ is in $\Omega(\ f(n)\ )$ if there exist constants c and n_0 such that $g(n) \geq c\, f(n)$ for all $n \geq n_0$
- Tight bound: $\theta(\ f(n)\ )$ is the set of all functions asymptotically equal to $f(n)$
  - Intersection of $O(\ f(n)\ )$ and $\Omega(\ f(n)\ )$ (use *different* c values)

Correct terms, in theory

A common error is to say $O(\ f(n)\ )$ when you mean $\theta(\ f(n)\ )$

- Since a linear algorithm is also $O(n^5)$, it's tempting to say "this algorithm is exactly $O(n)$"
- That doesn't mean anything, say it is $\theta(n)$
- That means that it is not, for example $O(\texttt{log}\ n)$

Less common notation:

- "little-oh": intersection of "big-Oh" and *not* "big-Theta"
  - *For all c*, there exists an n_0 such that… ≤
  - Example: array sum is $o(n^2)$ but not $o(n)$
- "little-omega": intersection of "big-Omega" and *not* "big-Theta"
  - *For all c*, there exists an n_0 such that… ≥
  - Example: array sum is $\omega(\texttt{log}\ n)$ but not $\omega(n)$

What we are analyzing

- The most common thing to do is give an O or θ bound to the worst-case running time of an algorithm

- Example: binary-search algorithm
  - Common: $\theta($`log` $n)$ running-time in the worst-case
  - Less common: $\theta(1)$ in the best-case (item is in the middle)
  - Less common: Algorithm is $\Omega($`log log` $n)$ in the worst-case (it is not really, really, really fast asymptotically)
  - Less common (but very good to know): the find-in-sorted-array ***problem*** is $\Omega($`log` $n)$ in the worst-case
    - *No* algorithm can do better
    - A ***problem*** cannot be $O(f(n))$ since you can always find a slower algorithm, but can mean ***there exists*** an algorithm

Other things to analyze

- Space instead of time
  - Remember we can often use space to gain time

- Average case
  - Sometimes only if you assume something about the *probability distribution* of inputs
  - Sometimes uses randomization in the algorithm
    - Will see an example with sorting
  - Sometimes an *amortized guarantee*
    - Average time over any sequence of operations
    - Will discuss in a later lecture

Summary

Analysis can be about:

- The problem or the algorithm (usually algorithm)
- Time or space (usually time)
  - Or power or dollars or …
- Best-, worst-, or average-case (usually worst)
- Upper-, lower-, or tight-bound (usually upper or tight)

Usually asymptotic is valuable

- Asymptotic complexity focuses on behavior for large *n* and is independent of any computer / coding trick

- But you can "abuse" it to be misled about trade-offs

- Example: $n^{1/10}$ vs. **`log`** *n*
  – Asymptotically $n^{1/10}$ grows more quickly
  – But the "cross-over" point is around $5 * 10^{17}$
  – So if you have input size less than 2^{58}, prefer $n^{1/10}$

- For *small n*, an algorithm with worse asymptotic complexity might be faster
  – Here the constant factors can matter, if you care about performance for small *n*

Timing vs. Big-Oh Summary

- Big-oh is an essential part of computer science's mathematical foundation
  - Examine the algorithm itself, not the implementation
  - Reason about (even prove) performance as a function of n

- Timing also has its place
  - Compare implementations
  - Focus on data sets you care about (versus worst case)
  - Determine what the constant factors "really are"