CSE373: Data Structures and Algorithms

Lecture 2: Math Review; Algorithm Analysis

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Today

- Any questions on stacks and queues?

- Review math essential to algorithm analysis
  - Proof by induction
  - Powers of 2
  - Binary numbers
  - Exponents and logarithms

- Begin analyzing algorithms
  - Using asymptotic analysis (continue next time)

Mathematical induction

Suppose *P(n)* is some predicate (mentioning integer *n*)

- Example: $n \geq n/2 + 1$

To prove *P(n)* for all integers $n \geq n_0$, it suffices to prove

1. $P(n_0)$ – called the "basis" or "base case"
2. If *P(k),* then *P(k+1)* – called the "induction step" or "inductive case"

Why we will care:

To show an algorithm is correct or has a certain running time *no matter how big a data structure or input value is*

(Our "*n*" will be the data structure or input size.)

Example

P(n) = "the sum of the first *n* powers of 2 (starting at 0) is 2^n-1"

Theorem: *P(n)* holds for all $n \geq 1$

Proof: By induction on *n*

- Base case: *n*=1. Sum of first 1 power of 2 is 2^0 , which equals 1.
  And for *n*=1, 2^n-1 equals 1.
- Inductive case:
  - Assume the sum of the first *k* powers of 2 is 2^k-1
  - Show the sum of the first (*k*+1) powers of 2 is 2^{k+1}-1

  Using assumption, sum of the first (*k*+1) powers of 2 is

  $(2^k-1) + 2^{(k+1)-1} = (2^k-1) + 2^k = 2^{k+1}-1$

Example

n	1	2	3	4
sum of first n powers of 2	$2^0 = 1$	$1 + 2^1 = 3$	$3 + 2^2 = 7$	$7 + 2^3 = 15$
P(n)	$2^1 - 1 = 1$			

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Example

n	1	2	3	4	
sum of first n powers of 2	$2^0 = 1$	$1 + 2^1 = 3$	$3 + 2^2 = 7$	$7 + 2^3 = 15$	…
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Powers of 2

- A bit is 0 or 1 (just two different “letters” or “symbols”)
- A sequence of *n* bits can represent 2^n distinct things
  - For example, the numbers 0 through 2^n-1
- 2^{10} is 1024 (“about a thousand”, kilo in CSE speak)
- 2^{20} is “about a million”, mega in CSE speak
- 2^{30} is “about a billion”, giga in CSE speak

Java: an **`int`** is 32 bits and signed, so “max int” is “about 2 billion”

a **`long`** is 64 bits and signed, so “max long” is 2^{63}-1

Therefore…

Could give a unique id to…

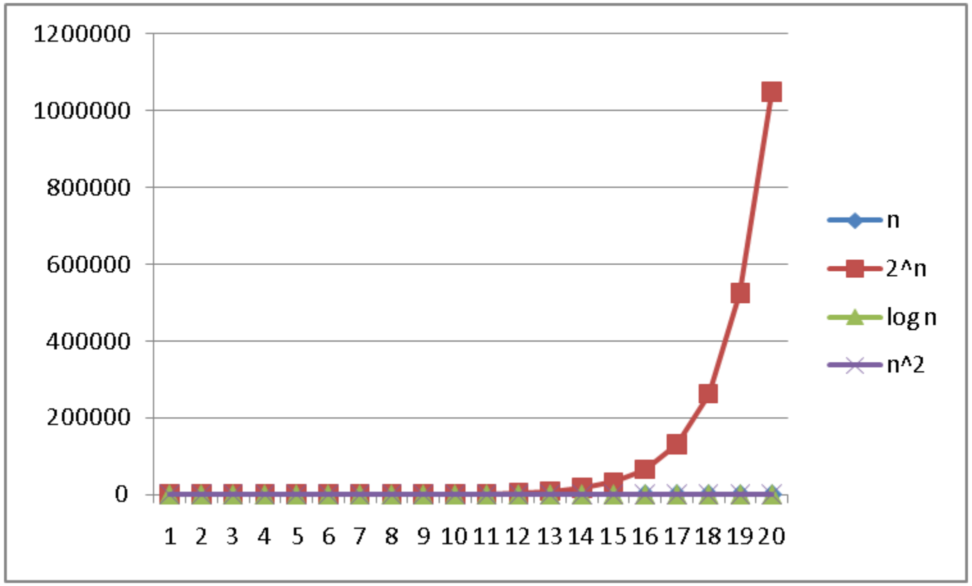
- Every person in the U.S. with 29 bits
- Every person in the world with 33 bits
- Every person to have ever lived with 38 bits (estimate)
- Every atom in the universe with 250-300 bits

So if a password is 128 bits long and randomly generated, do you think you could guess it?

Logarithms and Exponents

- Since so much is binary in CS **`log`** almost always means $\mathbf{log_2}$
- Definition: $\log_2 x = y$ if $x = 2^y$
- So, $\log_2$ 1,000,000 = "a little under 20"
- Just as exponents grow *very* quickly, logarithms grow *very* slowly

See Excel file for plot data – play with it!



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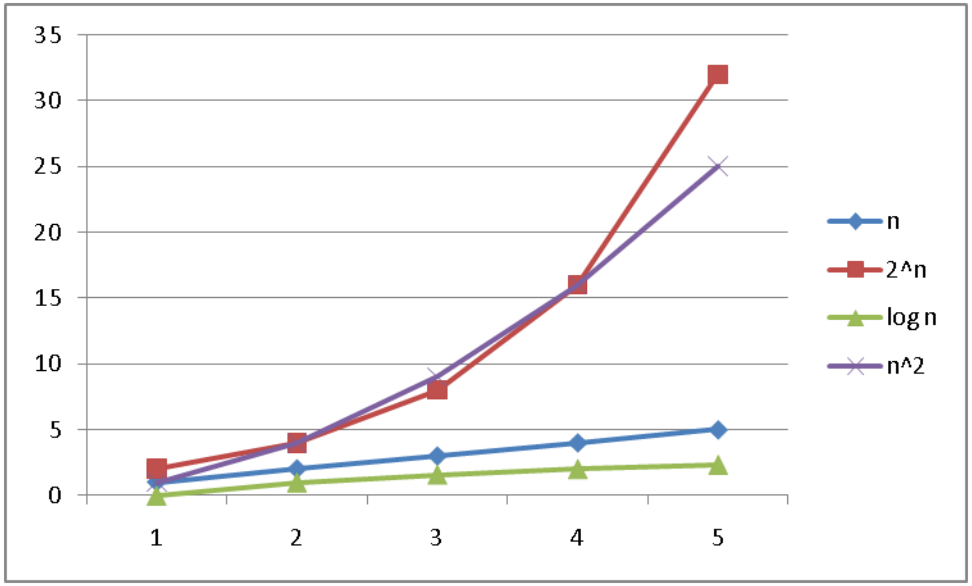

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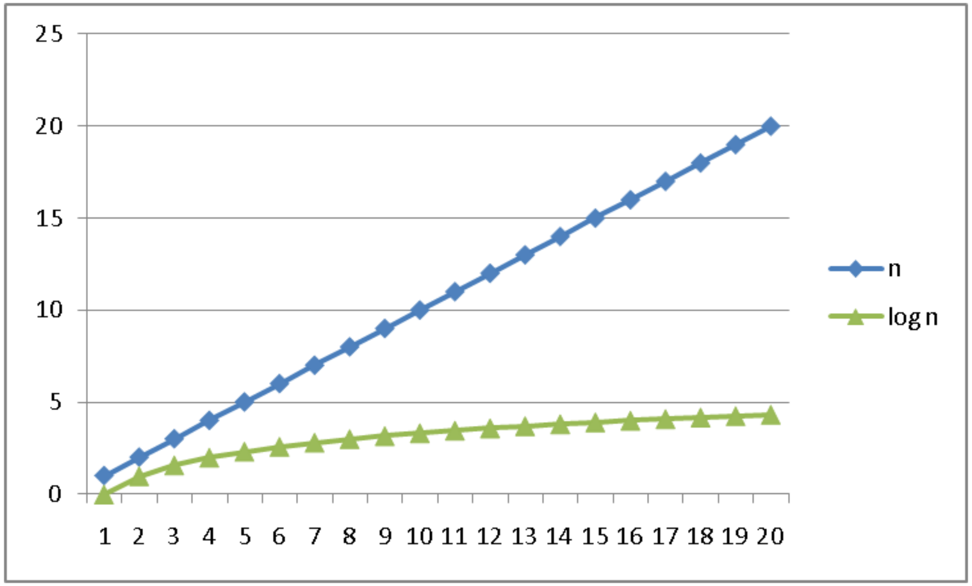

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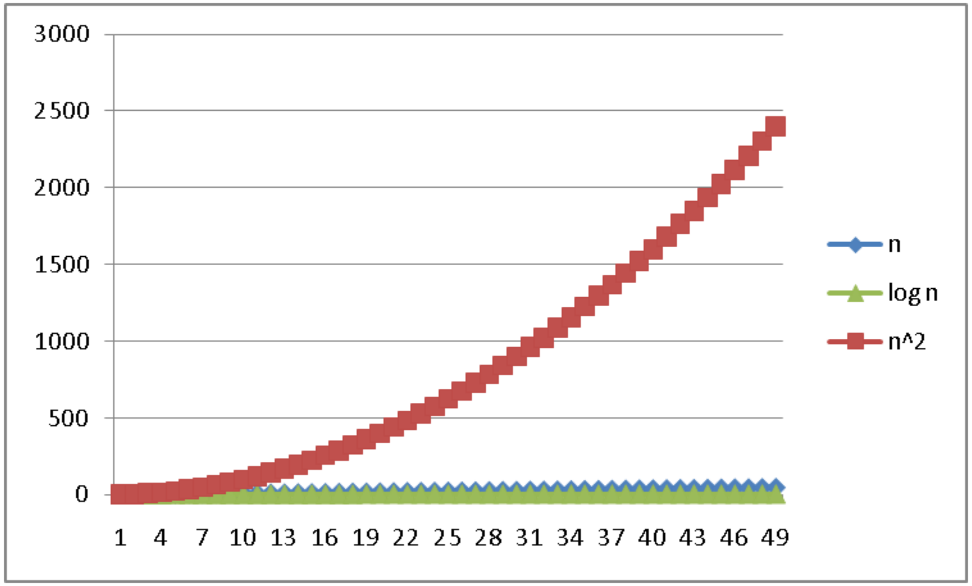


Properties of logarithms

- **`log(A*B) = log A + log B`**
  - So $\log(N^k) = k \log N$
- **`log(A/B) = log A – log B`**
- **`log(log x)`** is written **`log log x`**
  - Grows as slowly as 2^{2^y} grows quickly
- **`(log x)(log x)`** is written $\log^2 x$
  - It is greater than **`log x`** for all **`x > 2`**
  - It is not the same as **`log log x`**

Log base doesn't matter much!

"Any base *B* log is equivalent to base 2 log within a constant factor"

- And we are about to stop worrying about constant factors!
- In particular, $\log_2 x = 3.22 \log_{10} x$
- In general,

$$\log_B x = (\log_A x) / (\log_A B)$$

Floor and ceiling

$\lfloor X \rfloor$ Floor function: the largest integer $\leq$ X

$$\lfloor 2.7 \rfloor = 2 \qquad \lfloor -2.7 \rfloor = -3 \qquad \lfloor 2 \rfloor = 2$$

$\lceil X \rceil$ Ceiling function: the smallest integer $\geq$ X

$$\lceil 2.3 \rceil = 3 \qquad \lceil -2.3 \rceil = -2 \qquad \lceil 2 \rceil = 2$$

Floor and ceiling properties

1. $X - 1 < \lfloor X \rfloor \leq X$
2. $X \leq \lceil X \rceil < X + 1$
3. $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n \quad \text{if } n \text{ is an integer}$

Algorithm Analysis

As the “size” of an algorithm’s input grows (integer, length of array, size of queue, etc.):

- How much longer does the algorithm take (time)
- How much more memory does the algorithm need (space)

Because the curves we saw are so different, often care about only “which curve we are like”

Separate issue: Algorithm *correctness* – does it produce the right answer for all inputs

- Usually more important, naturally

Example

- What does this pseudocode return?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- Correctness: For any N ≥ 0, it returns…

Example

- What does this pseudocode return?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- Correctness: For any N ≥ 0, it returns 3N(N+1)/2
- Proof: By induction on *n*
  - *P(n)* = after outer for-loop executes *n* times, **`x`** holds 3n(n+1)/2
  - Base: n=0, returns 0
  - Inductive: From *P(k)*, **`x`** holds 3k(k+1)/2 after *k* iterations. Next iteration adds 3(k+1), for total of 3k(k+1)/2 + 3(k+1) = (3k(k+1) + 6(k+1))/2 = (k+1)(3k+6)/2 = 3(k+1)(k+2)/2

Example

- How long does this pseudocode run?

```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- Running time: For any N ≥ 0,
  - Assignments, additions, returns take “1 unit time”
  - Loops take the sum of the time for their iterations

- So: 2 + 2*(number of times inner loop runs)
  - And how many times is that…

Example

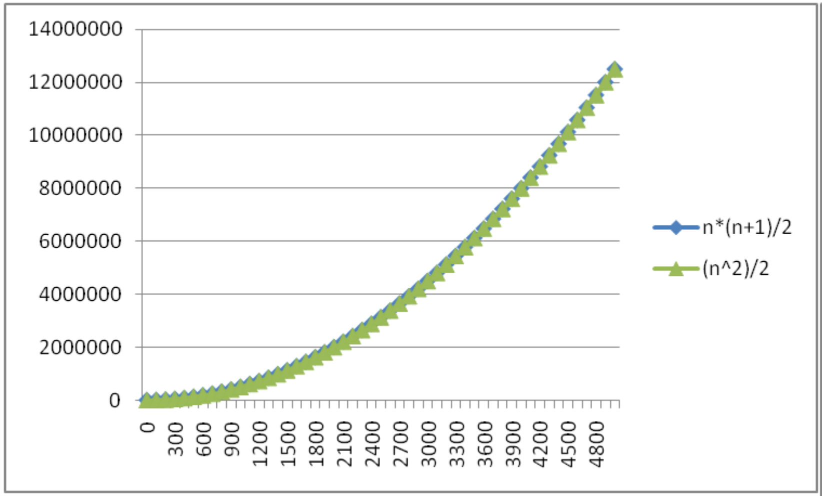
- How long does this pseudocode run?

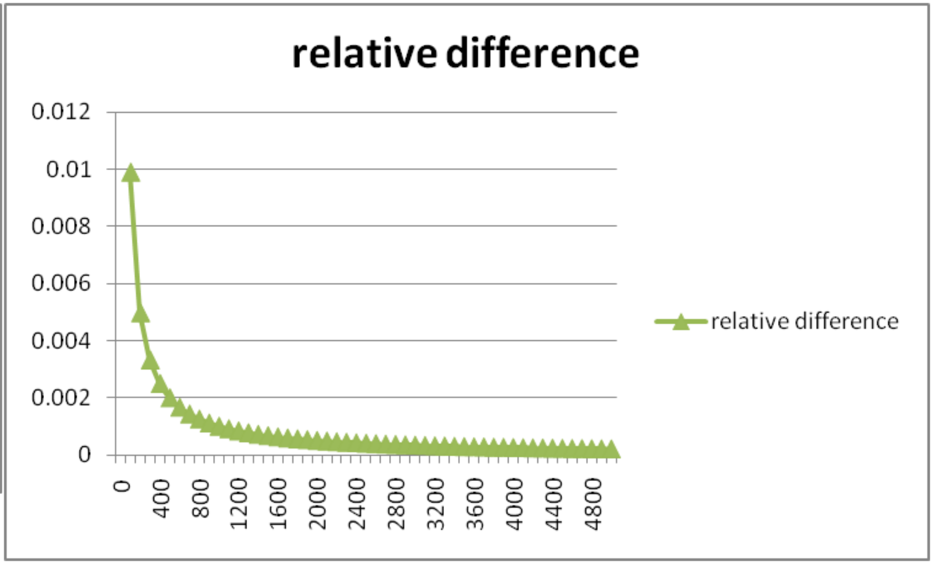
```
x := 0;
for i=1 to N do
  for j=1 to i do
     x := x + 3;
return x;
```

- The total number of loop iterations is N*(N+1)/2
  - This is a very common loop structure, worth memorizing
  - Proof is by induction on N, known for centuries
  - This is *proportional to* N^2 , and we say $O(N^2)$, “big-Oh of”
    - For large enough N, the N and constant terms are irrelevant, as are the first assignment and return
    - See plot… N*(N+1)/2 vs. just $N^2/2$

Lower-order terms don't matter

N*(N+1)/2 vs. just $N^2/2$

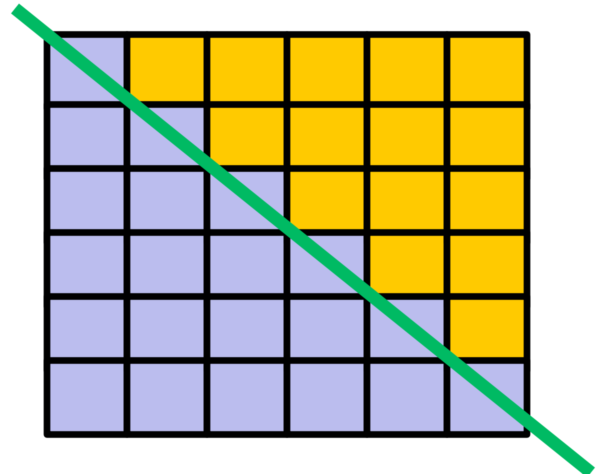




Geometric interpretation

$$\sum_{i=1}^{N} i = N*N/2+N/2$$

```
for i=1 to N do
  for j=1 to i do
     // small work
```



- Area of square: N*N
- Area of lower triangle of square: N*N/2
- Extra area from squares crossing the diagonal: N*1/2
- As N grows, fraction of “extra area” compared to lower triangle goes to zero (becomes insignificant)

Big-O: Common Names

$O(1)$	constant (same as $O(k)$ for constant k)
$O(\texttt{log}\ n)$	logarithmic
$O(n)$	linear
$O(\mathrm{n}\ \texttt{log}\ n)$	"n **`log`** *n*"
$O(n^2)$	quadratic
$O(n^3)$	cubic
$O(n^k)$	polynomial (where is k is any constant)
$O(k^n)$	exponential (where k is any constant > 1)

Note: "exponential" does not mean "grows really fast", it means "grows at rate proportional to k^n for some $k>1$"

- A savings account accrues interest exponentially (k=1.01?)
- If you don't know k, you probably don't know it's exponential