CSE373: Data Structures & Algorithms
Lecture 11: Implementing Union-Find

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Extra office hours

- Tuesday, 4:30-5:30, Bagley 154
- Thursday, 4:30-5:30, Bagley 154

Union-Find

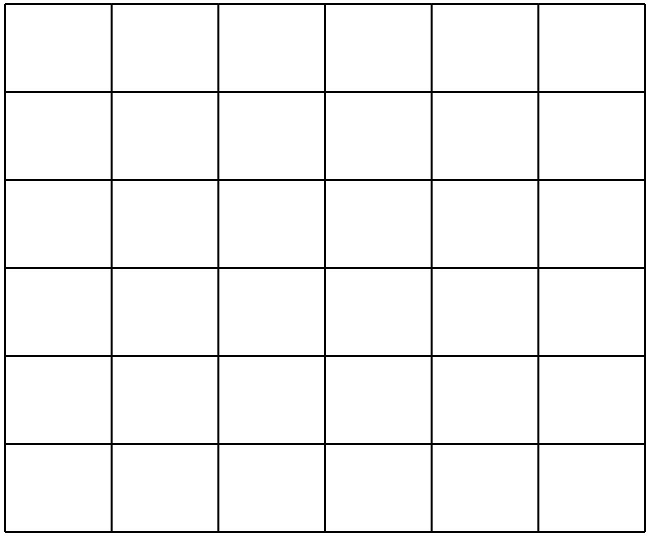
- Given an unchanging set *S*, **`create`** an initial partition of a set
  - Typically each item in its own subset: {a}, {b}, {c}, …
  - Give each subset a “name” by choosing a *representative element*

- Operation **`find`** takes an element of *S* and returns the representative element of the subset it is in

- Operation **`union`** takes two subsets and (permanently) makes one larger subset
  - A different partition with one fewer set
  - Affects result of subsequent **`find`** operations
  - Choice of representative element up to implementation

Example application: maze-building

- Build a random maze by erasing edges

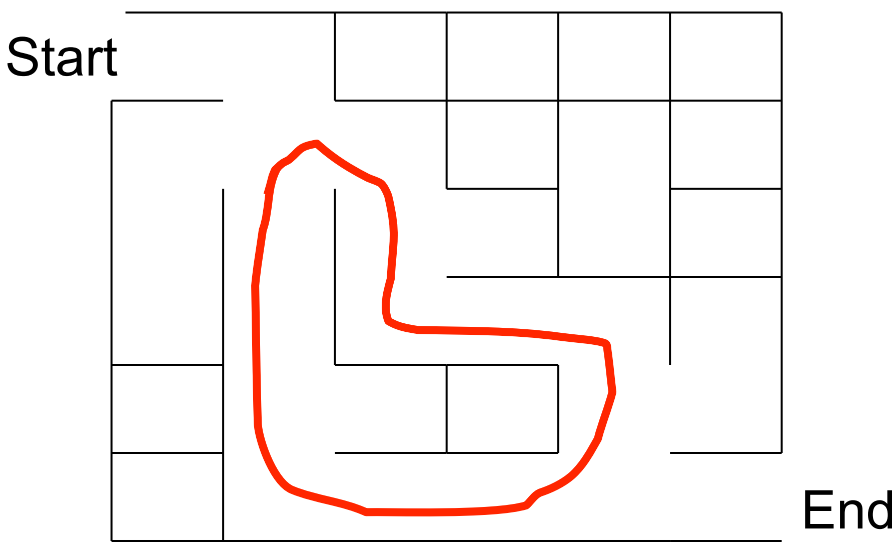


  – Possible to get from anywhere to anywhere
    - Including “start” to “finish”
  – No loops possible without backtracking
    - After a “bad turn” have to “undo”

Problems with this approach

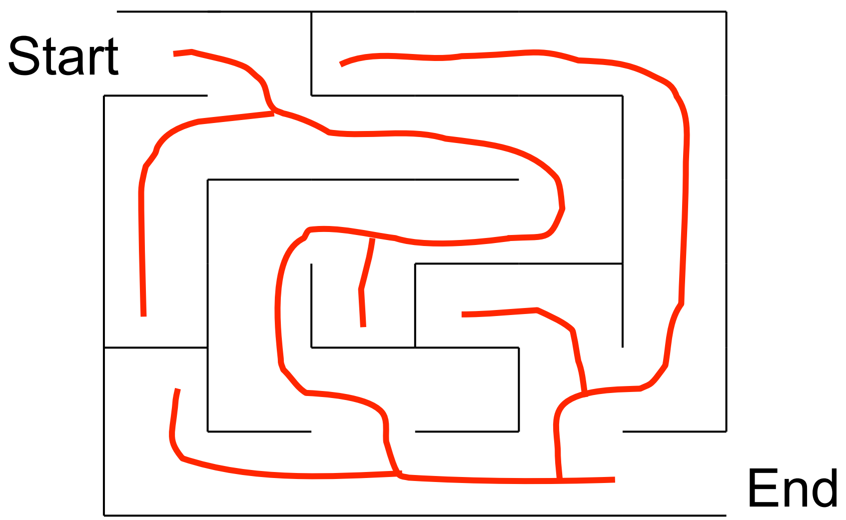
1. How can you tell when there is a path from start to finish?
   - We do not really have an algorithm yet

2. We have *cycles*, which a "good" maze avoids
   - Want one solution and no cycles



Revised approach

- Consider edges in random order
- But only delete them if they introduce no cycles (how? TBD)
- When done, will have one way to get from any place to any other place (assuming no backtracking)



- Notice the funny-looking *tree* in red

Cells and edges

- Let's number each cell
  - 36 total for 6 x 6
- An (internal) edge (x,y) is the line between cells x and y
  - 60 total for 6x6: (1,2), (2,3), …, (1,7), (2,8), …

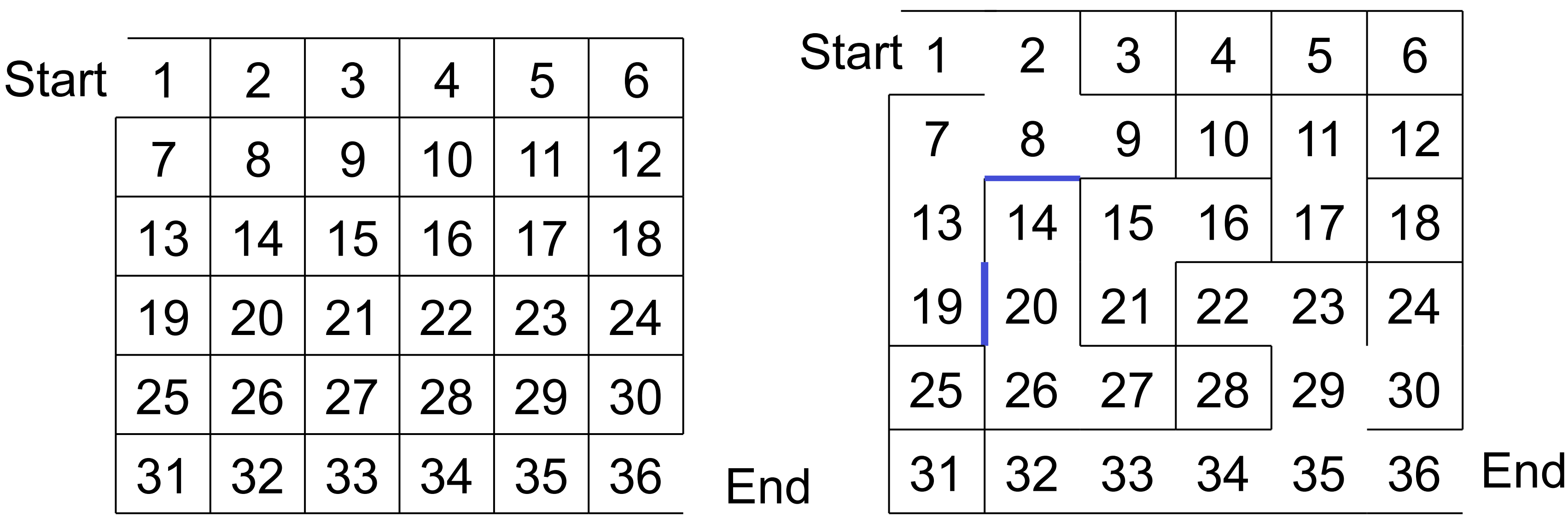
Start

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

The trick

- Partition the cells into **disjoint sets**: “are they connected”
  - Initially every cell is in its own subset
- If an edge would connect two different subsets:
  - then remove the edge and **union** the subsets
  - else leave the edge because removing it makes a cycle



The algorithm

- P = **disjoint sets** of connected cells, initially each cell in its own 1-element set
- E = **set** of edges not yet processed, initially all (internal) edges
- M = **set** of edges kept in maze (initially empty)

while P has more than one set {

- Pick a random edge (x,y) to remove from E
- u = **`find`**(x)
- v = **`find`**(y)
- if u==v

  then add (x,y) to M // same subset, do not create cycle

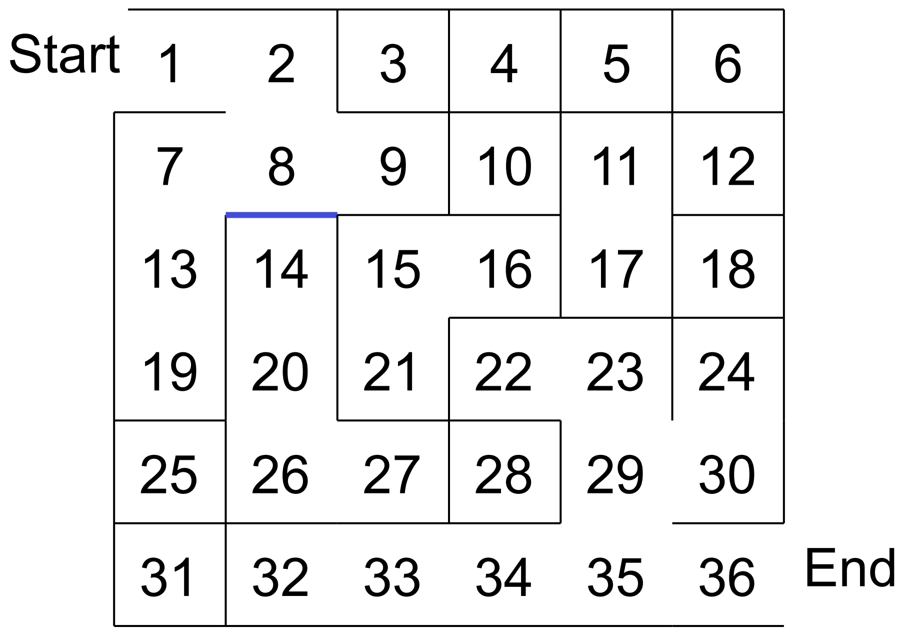
  else **`union`**(u,v) // do not put edge in M, connect subsets

}

Add remaining members of E to M, then output M as the maze

Example step

Pick (8,14)



P

{1,2,7,8,9,13,19}

{3}

{4}

{5}

{6}

{10}

{11,17}

{12}

{14,20,26,27}

{15,16,21}

{18}

{25}

{28}

{31}

{22,23,24,29,30,32
33,34,35,36}

Example step

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

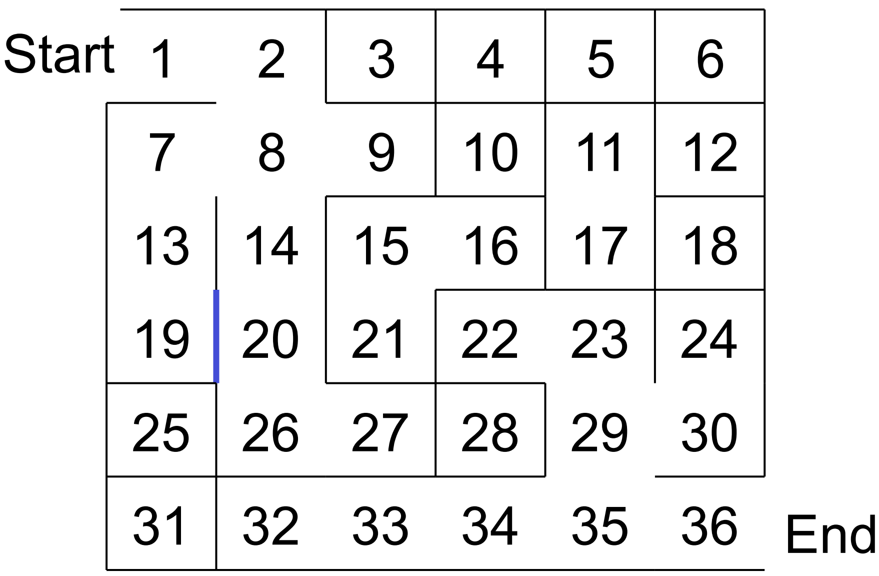
Find(8) = 7
Find(14) = 20

Union(7,20)

P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

Add edge to M step

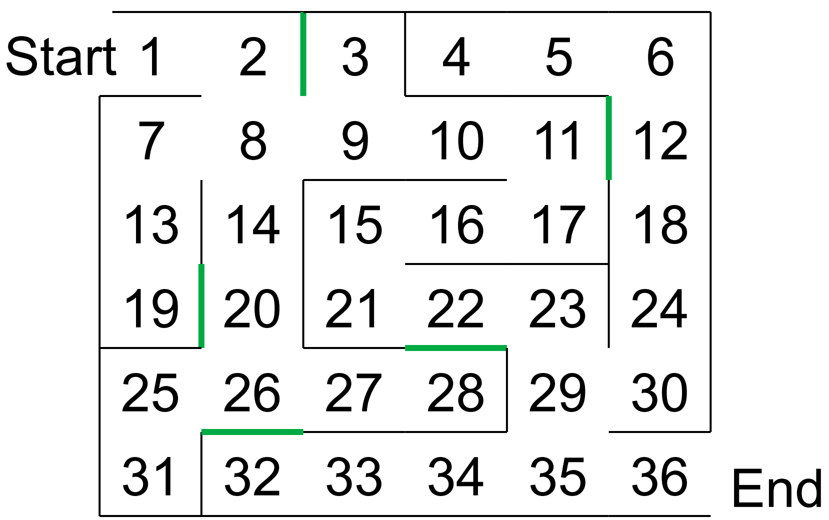
Pick (19,20)



P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

At the end

- Stop when P has one set
- Suppose green edges are already in M and black edges were not yet picked
  - Add all black edges to M



P
{1,2,3,4,5,6,7,… 36}

Other applications

- Maze-building is:
  - Cute
  - Homework 4 ☺
  - A surprising use of the union-find ADT
- Many other uses (which is why an ADT taught in CSE373):
  - Road/network/graph connectivity (will see this again)
    - "connected components" e.g., in social network
  - Partition an image by connected-pixels-of-similar-color
  - Type inference in programming languages
- Not as common as dictionaries, queues, and stacks, but valuable because implementations are very fast, so when applicable can provide big improvements

The plan

Last lecture:

- What are *disjoint sets*
  - And how are they "the same thing" as *equivalence relations*
- The union-find ADT for disjoint sets

Now:

- Applications of union-find
- Basic implementation of the ADT with "up trees"
- Optimizations that make the implementation much faster

Implementation – our goal

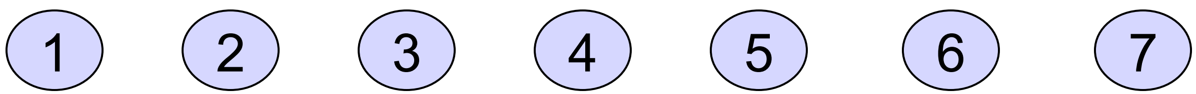
- Start with an initial partition of *n* subsets
  - Often 1-element sets, e.g., {1}, {2}, {3}, …, {*n*}

- May have *m* **`find`** operations and up to *n*-1 **`union`** operations in any order
  - After *n*-1 **`union`** operations, every **`find`** returns same 1 set

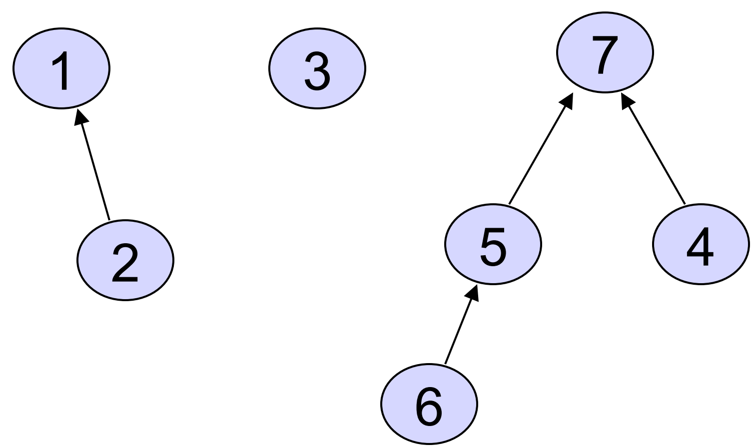
- If total for all these operations is *O*(*m*+*n*), then amortized *O*(1)
  - We will get very, very close to this
  - *O*(1) worst-case is impossible for **`find`** *and* **`union`**
    - Trivial for one *or* the other

Up-tree data structure

- Tree with:
  - No limit on branching factor
  - References from children to parent
- Start with *forest* of 1-node trees



- Possible forest after several unions:
  - Will use roots for set names

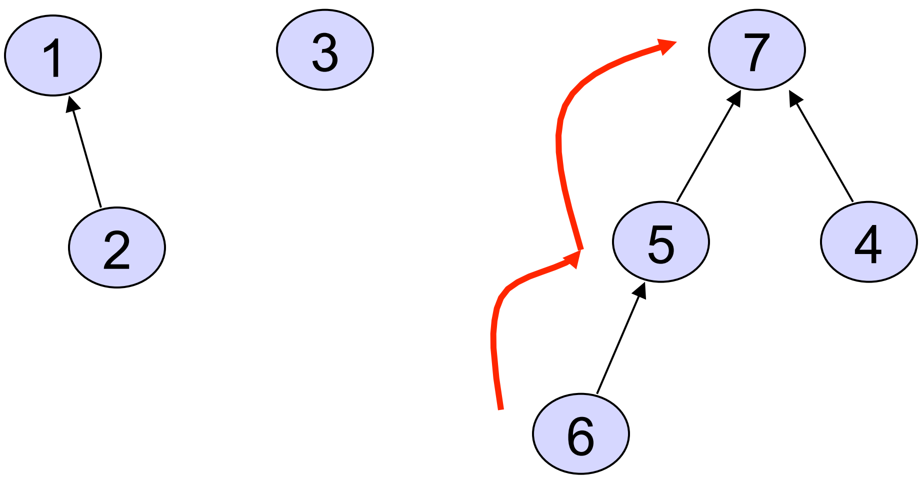


Find

`find`(**`x`**):

- *Assume* we have *O*(1) access to each node
  - Will use an array where index **`i`** holds node **`i`**
- Start at **`x`** and follow parent pointers to root
- Return the root

`find`(6) = 7

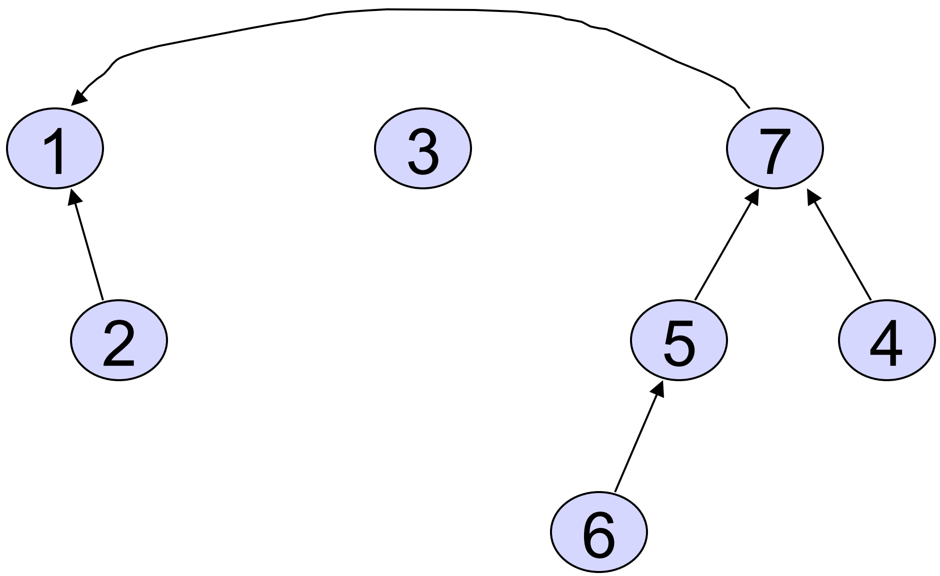


Union

`union(x,y)`:

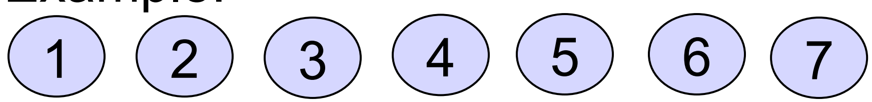
- Assume **`x`** and **`y`** are roots
  - Else find the roots of their trees
- Assume distinct trees (else do nothing)
- Change root of one to have parent be the root of the other
  - Notice no limit on branching factor

`union`(1,7)



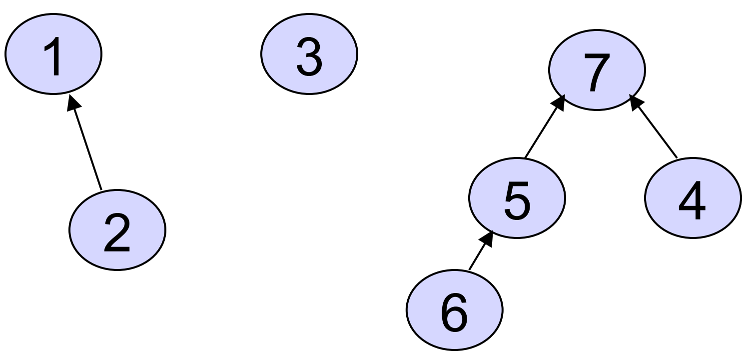
Simple implementation

- If set elements are contiguous numbers (e.g., 1,2,…,*n*), use an array of length *n* called **`up`**
  - Starting at index 1 on slides
  - Put in array index of parent, with 0 (or -1, etc.) for a root
- Example:



	1	2	3	4	5	6	7
up	0	0	0	0	0	0	0

- Example:



	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0

- If set elements are not contiguous numbers, could have a separate dictionary to map elements (keys) to numbers (values)

Implement operations

```
// assumes x in range 1,n
int find(int x) {
  while(up[x] != 0) {
     x = up[x];
  }
  return x;
}
```

```
// assumes x,y are roots
void union(int x, int y){
  up[y] = x;
}
```

- Worst-case run-time for **`union`**?
- Worst-case run-time for **`find`**?
- Worst-case run-time for *m* **`find`**s and *n*-1 **`union`**s?

Implement operations

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- Worst-case run-time for **`union`**? *O*(1)
- Worst-case run-time for **`find`**?
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Implement operations

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  while(up[x] != 0) {
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```

```
// assumes x,y are roots
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```

- Worst-case run-time for **`union`**? *O*(1)
- Worst-case run-time for **`find`**? *O*(*n*)
- Worst-case run-time for *m* **`find`**s and *n*-1 **`union`**s?

Implement operations

```
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- Worst-case run-time for **`union`**? $O(1)$
- Worst-case run-time for **`find`**? $O(n)$
- Worst-case run-time for *m* **`find`**s and *n*-1 **`union`**s? $O(n*m)$

The plan

Last lecture:

- What are *disjoint sets*
  - And how are they "the same thing" as *equivalence relations*
- The union-find ADT for disjoint sets
- Applications of union-find

Now:

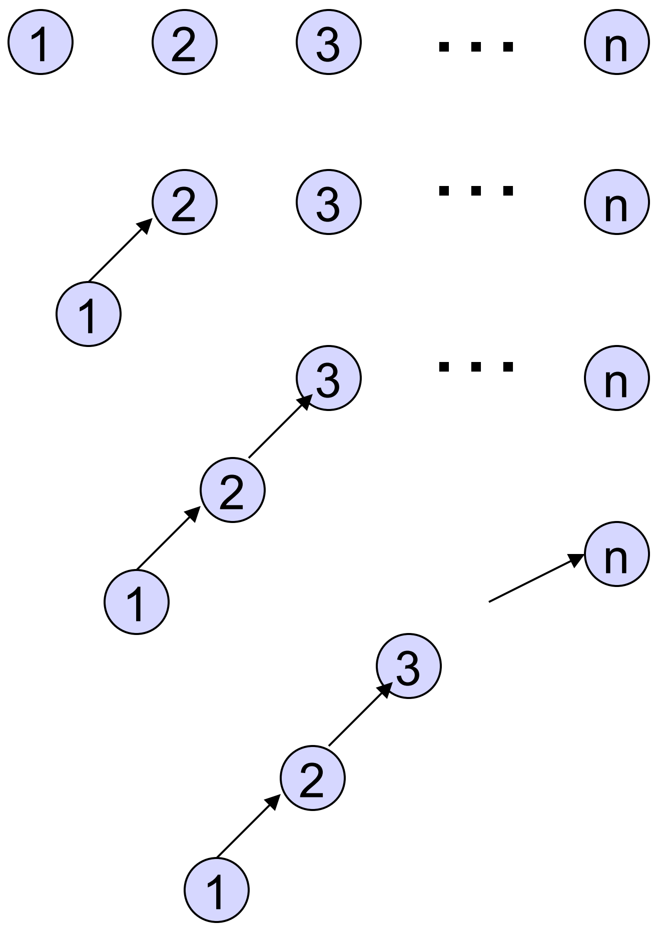
- Basic implementation of the ADT with "up trees"
- Optimizations that make the implementation much faster

Two key optimizations

1. Improve **`union`** so it stays *O(1)* but makes **`find`** $O(\texttt{log}\ n)$
   - So *m* **`find`**s and *n*-1 **`union`**s is $O(m \texttt{ log } n + n)$
   - *Union-by-size:* connect smaller tree to larger tree

2. Improve **`find`** so it becomes even faster
   - Make *m* **`find`**s and *n*-1 **`union`**s ***almost*** $O(m + n)$
   - *Path-compression:* connect directly to root during finds

The bad case to avoid



union(2,1)

union(3,2)

:

.

union(n,n-1)

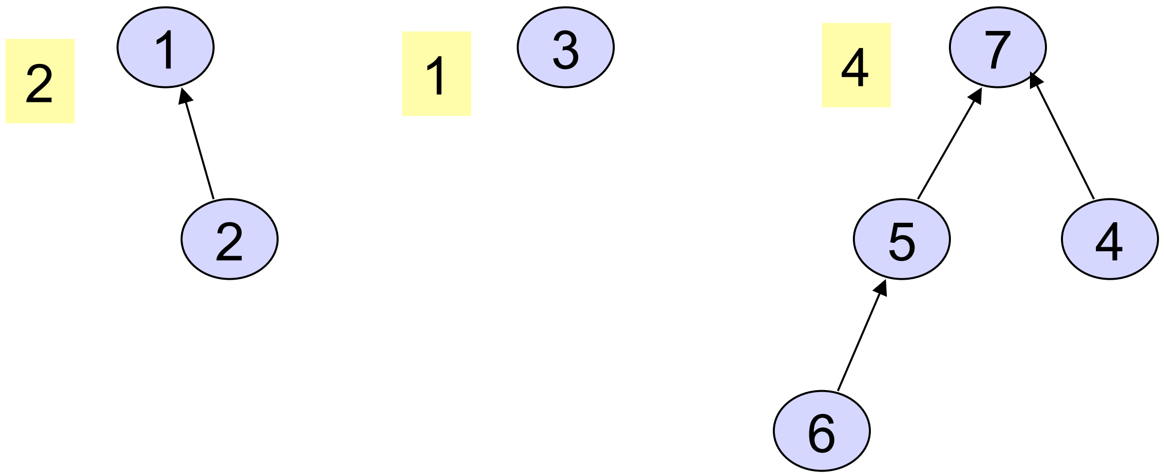
find(1) n steps!!

Union-by-size

Union-by-size:

- Always point the *smaller* (total # of nodes) tree to the root of the larger tree

`union`(1,7)



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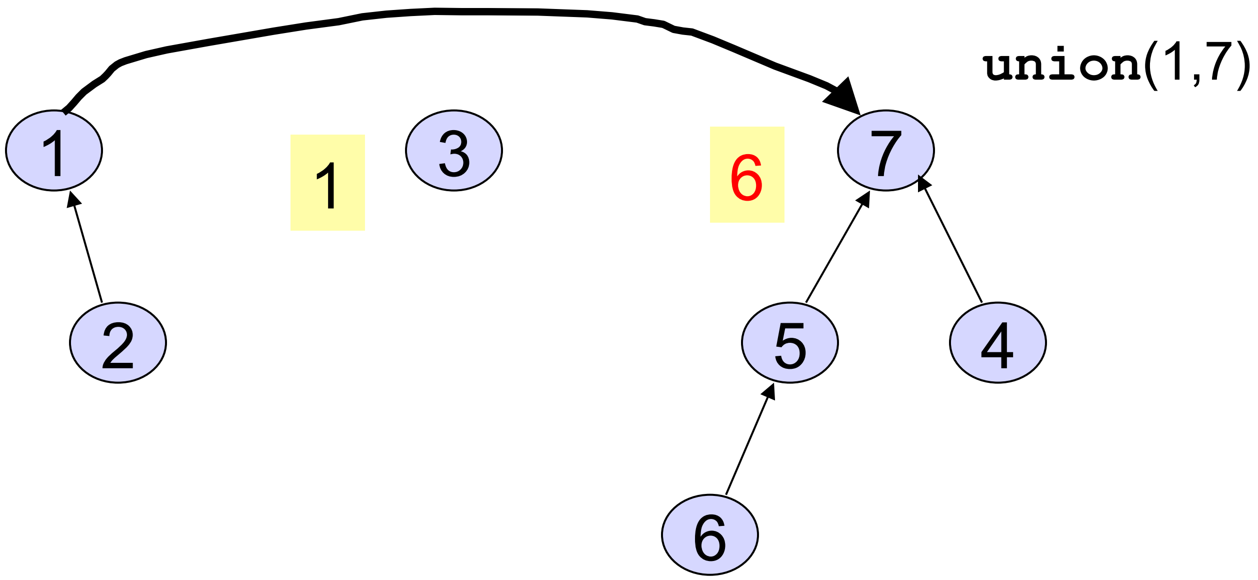
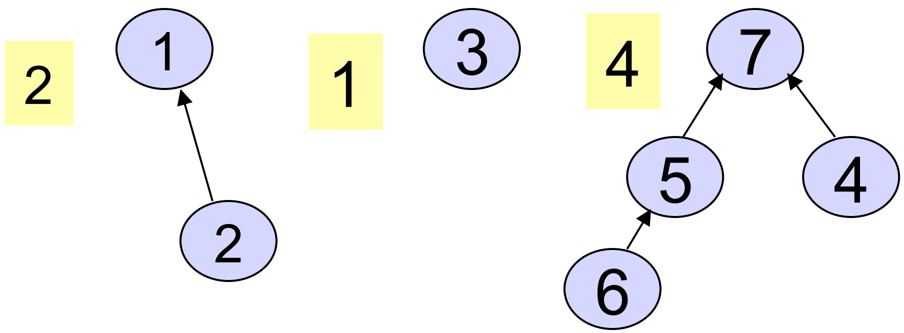


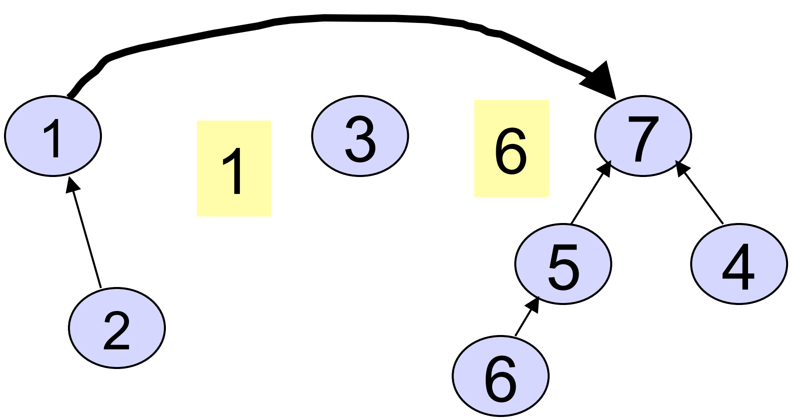
Array implementation

Keep the size (number of nodes in a second array)

- Or have one array of objects with two fields



	1	2	3	4	5	6	7
up	0	1	0	7	7	5	0
weight	2		1				4

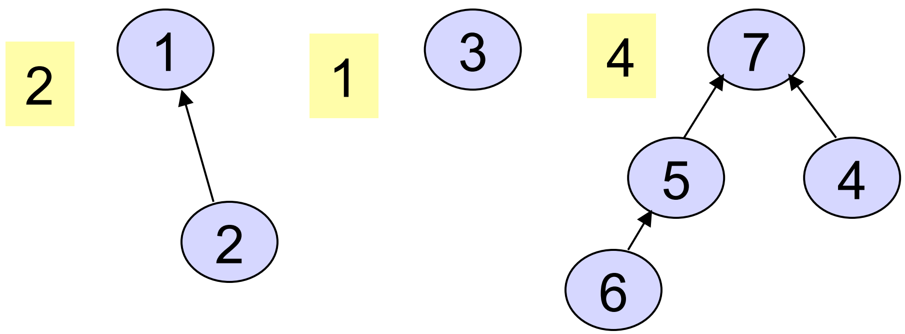


up	7	1	0	7	7	5	0
weight	2		1				6

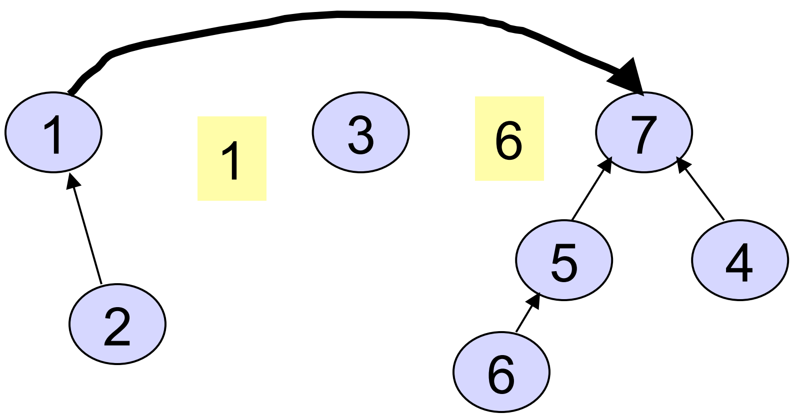
Nifty trick

Actually we do not need a second array…

- Instead of storing 0 for a root, store negation of size
- So up value < 0 means a root

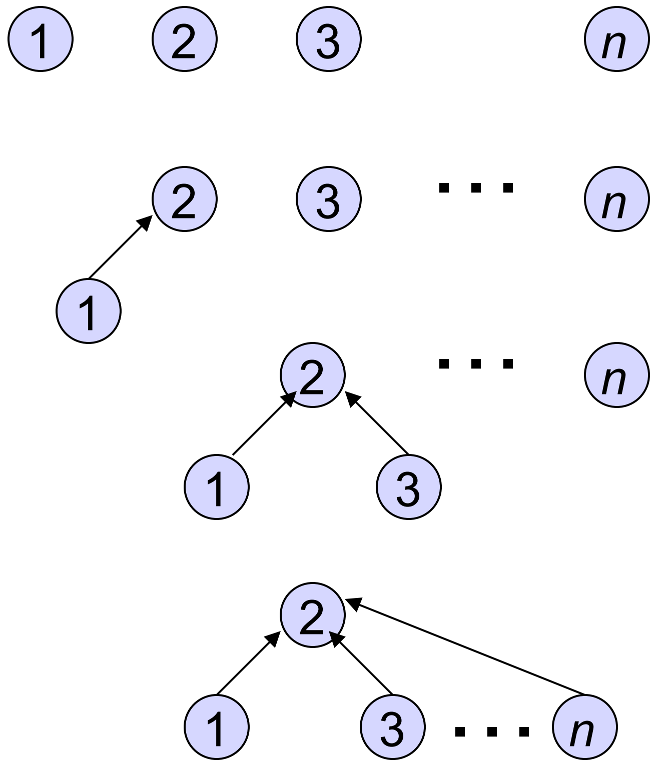


	1	2	3	4	5	6	7
up	-2	1	-1	7	7	5	-4



up	7	1	-1	7	7	5	-6

Bad example? Great example…



`union`(2,1)

`union`(3,2)

:

`union`(n,n-1)

`find`(1) *constant here*

General analysis

- Showing one worst-case example is now good is *not* a proof that the worst-case has improved

- So let's prove:
  - **`union`** is still $O(1)$ – this is "obvious"
  - **`find`** is now $O(\texttt{log}\ n)$

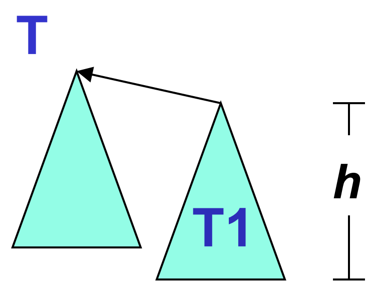
- Claim: If we use union-by-size, an up-tree of height h has at least 2^h nodes
  - Proof by induction on h…

Exponential number of nodes

P(h)= With union-by-size, up-tree of height h has at least 2^h nodes

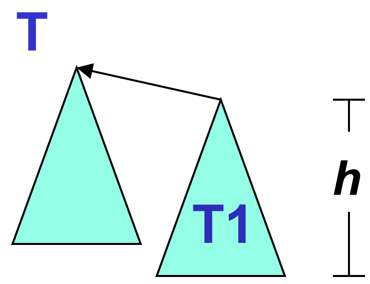
Proof by induction on h…

- Base case: $h = 0$: The up-tree has 1 node and $2^0 = 1$
- Inductive case: Assume P(h) and show P(h+1)
  - A height h+1 tree T has at least one height h child T1
  - T1 has at least 2^h nodes by induction
  - And T has *at least* as many nodes not in T1 than in T1
    - Else union-by-size would have had T point to T1, not T1 point to T (!!)
  - So total number of nodes is *at least* $2^h + 2^h = 2^{h+1}$



The key idea

Intuition behind the proof: No one child can have more than half the nodes

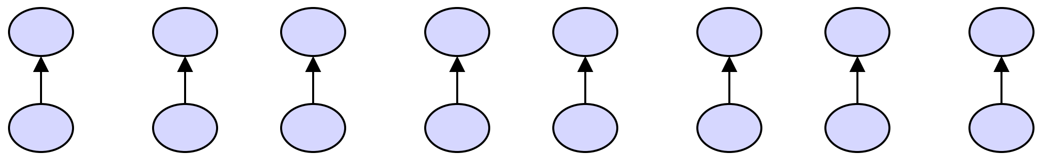


So, as usual, if number of nodes is exponential in height, then height is logarithmic in number of nodes

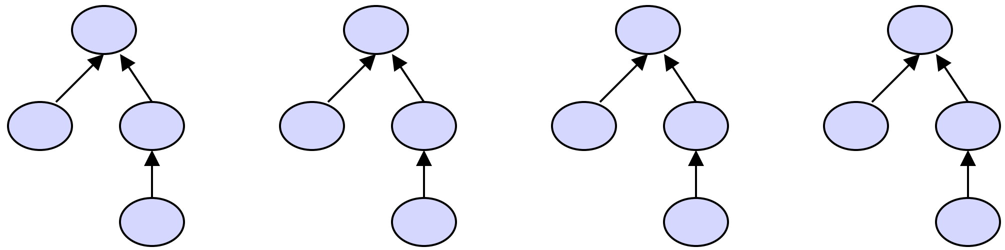
So **`find`** is $O($**`log`** $n)$

The new worst case

n/2 Unions-by-size

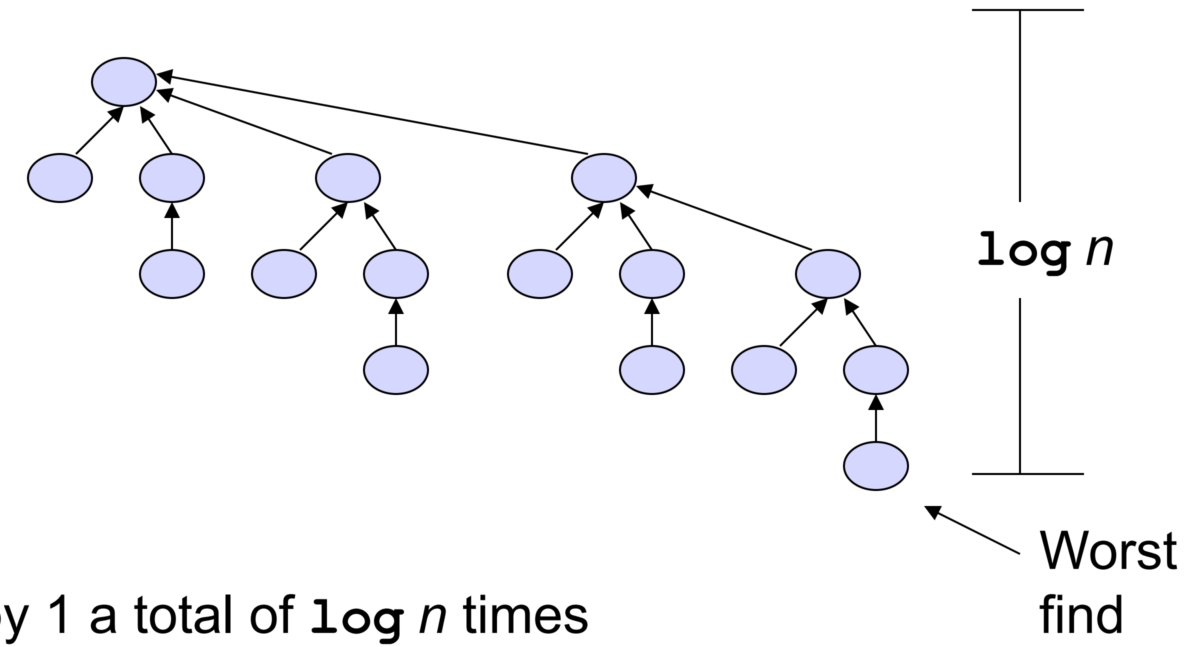


n/4 Unions-by-size



The new worst case (continued)

After n/2 + n/4 + …+ 1 Unions-by-size:



Height grows by 1 a total of **log** *n* times

What about union-by-height

We could store the height of each root rather than size

- Still guarantees logarithmic worst-case find
  - Proof left as an exercise if interested

- But does not work well with our next optimization
  - Maintaining height becomes inefficient, but maintaining size still easy

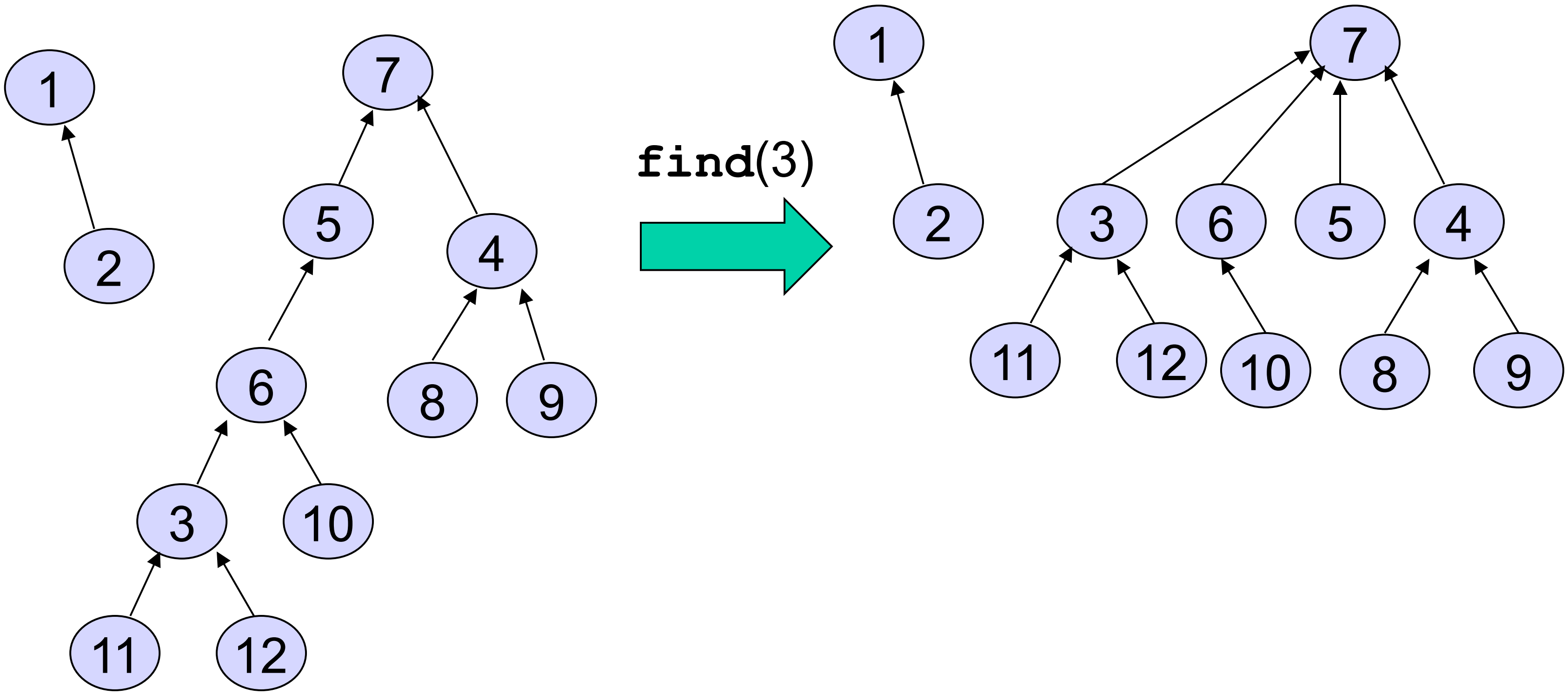
Two key optimizations

1. Improve **`union`** so it stays *O(1)* but makes **`find`** *O*(**`log`** *n*)
   - So *m* **`find`**s and *n*-1 **`union`**s is *O*(*m* **`log`** *n* + *n*)
   - *Union-by-size:* connect smaller tree to larger tree

2. Improve **`find`** so it becomes even faster
   - Make *m* **`find`**s and *n*-1 **`union`**s ***almost*** *O*(*m* + *n*)
   - *Path-compression:* connect directly to root during finds

Path compression

- Simple idea: As part of a **`find`**, change each encountered node's parent to point directly to root
  - Faster future **`find`**s for everything on the path (and their descendants)



Pseudocode

```
// performs path compression
int find(i) {
  // find root
  int r = i
  while(up[r] > 0)
     r = up[r]
  // compress path
  if i==r
    return r;
  int old_parent = up[i]
  while(old_parent != r) {
    up[i] = r
    i = old_parent;
    old_parent = up[i]
  }
  return r;
}
```

So, how fast is it?

A single worst-case **`find`** could be *O*(**`log`** *n*)

- But only if we did a lot of worst-case unions beforehand
- And path compression will make future finds faster

Turns out the amortized worst-case bound is much better than *O*(**`log`** *n*)

- We won't *prove* it – see text if curious
- But we will *understand* it:
  - How it is *almost* *O*(1)
  - Because total for *m* **`find`**s and *n*-1 **`union`**s is *almost* *O*(*m*+*n*)

A really slow-growing function

`log*` *x* is the minimum number of times you need to apply "**`log`** of **`log`** of **`log`** of" to go from *x* to a number <= 1

For just about every number we care about, **`log*`** *x* is *5* (!)

If $x <= 2^{65536}$ then **`log*`** $x <= 5$

- $\log^* 2 = 1$
- $\log^* 4 = \log^* 2^2 = 2$
- $\log^* 16 = \log^* 2^{(2^2)} = 3$ (log log log 16 = 1)
- $\log^* 65536 = \log^* 2^{((2^2)^2)} = 4$ (log log log log 65536 = 1)
- $\log^* 2^{65536} = \ldots\ldots\ldots\ldots\ldots = 5$

Almost linear

- Turns out total time for *m* **`find`**s and *n*-1 **`union`**s is $O((m+n)*(\textbf{log*}\ (m+n))$
  - Remember, if $m+n < 2^{65536}$ then **`log*`** $(m+n) < 5$

- At this point, it feels almost silly to mention it, but even that bound is not tight…
  - "Inverse Ackerman's function" grows even more slowly than **`log*`**
    - Inverse because Ackerman's function grows really fast
    - Function also appears in combinatorics and geometry
    - For any number you can possibly imagine, it is < 4
  - Can replace **`log*`** with "Inverse Ackerman's" in bound

Theory and terminology

- Because `log*` or Inverse Ackerman's grows soooo slowly
  - For all practical purposes, amortized bound is constant, i.e., total cost is linear
  - We say "near linear" or "effectively linear"

- Need union-by-size and path-compression for this bound
  - Path-compression changes height but not weight, so they interact well

- As always, asymptotic analysis is separate from "coding it up"