CSE373: Data Structures & Algorithms
Lecture 17: Shortest Paths

Nicki Dell
Spring 2014
Announcements

• Midterm on Friday May 9th
  – TAs proctoring the exam
  – Be on time!!

• Homework 4 partner selection due TODAY!
• Homework 4 due next Wednesday, May 14th
Graph Traversals

For an arbitrary graph and a starting node $v$, find all nodes reachable from $v$ (i.e., there exists a path from $v$).

Basic idea:
- Keep following nodes
- But “mark” nodes after visiting them, so the traversal terminates and processes each reachable node exactly once

Important Graph traversal algorithms:
- “Depth-first search” “DFS”: recursively explore one part before going back to the other parts not yet explored
- “Breadth-first search” “BFS”: explore areas closer to the start node first
Example: Another Depth First Search

• A tree is a graph and DFS and BFS are particularly easy to “see”

```java
DFS2(Node start) {
    initialize stack s and push start
    mark start as visited
    while(s is not empty) {
        next = s.pop() // and “process”
        for each node u adjacent to next
            if(u is not marked)
                mark u and push onto s
    }
}
```

• Could be other correct DFS traversals (e.g. go to right nodes first)
• The marking is because we support arbitrary graphs and we want to process each node exactly once
Example: Breadth First Search

- A tree is a graph and DFS and BFS are particularly easy to “see”

BFS(Node start) {
  initialize queue q and enqueue start
  mark start as visited
  while(q is not empty) {
    next = q.dequeue() // and “process”
    for each node u adjacent to next
      if(u is not marked)
        mark u and enqueue onto q
  }
}

- A B C D E F G H
- A “level-order” traversal
Comparison

• Breadth-first always finds shortest paths, i.e., “optimal solutions”
  – Better for “what is the shortest path from x to y”

• But depth-first can use less space in finding a path
  – If longest path in the graph is p and highest out-degree is d
    then DFS stack never has more than d*p elements
  – But a queue for BFS may hold O(|V|) nodes

• A third approach:
  – Iterative deepening (IDFS):
    • Try DFS but disallow recursion more than k levels deep
    • If that fails, increment k and start the entire search over
  – Like BFS, finds shortest paths. Like DFS, less space.
Saving the Path

• Our graph traversals can answer the reachability question:
  – “Is there a path from node x to node y?”

• But what if we want to actually output the path?
  – Like getting driving directions rather than just knowing it’s possible to get there!

• How to do it:
  – Instead of just “marking” a node, store the previous node along the path (when processing $u$ causes us to add $v$ to the search, set $v$.path field to be $u$)
  – When you reach the goal, follow path fields back to where you started (and then reverse the answer)
  – If just wanted path length, could put the integer distance at each node instead
Example using BFS

What is a path from Seattle to Tyler

- Remember marked nodes are not re-enqueued
- Note shortest paths may not be unique
Single source shortest paths

• Done: BFS to find the minimum path length from \( v \) to \( u \) in \( O(|E|+|V|) \)

• Actually, can find the minimum path length from \( v \) to every node
  – Still \( O(|E|+|V|) \)
  – No faster way for a “distinguished” destination in the worst-case

• Now: Weighted graphs

  Given a weighted graph and node \( v \),
  find the minimum-cost path from \( v \) to every node

• As before, asymptotically no harder than for one destination
Applications

• Driving directions

• Cheap flight itineraries

• Network routing

• Critical paths in project management
Not as easy as BFS

Why BFS won’t work: Shortest path may not have the fewest edges
  – Annoying when this happens with costs of flights

We will assume there are no negative weights
• *Problem* is *ill-defined* if there are negative-cost cycles
• *Today’s algorithm* is *wrong* if edges can be negative
  – There are other, slower (but not terrible) algorithms
Dijkstra’s Algorithm

• Named after its inventor Edsger Dijkstra (1930-2002)
  – Truly one of the “founders” of computer science; this is just one of his many contributions
  – Many people have a favorite Dijkstra story, even if they never met him

Computer science is no more about computers than astronomy is about telescopes.

(Edsger Dijkstra)
Dijkstra’s Algorithm

- The idea: reminiscent of BFS, but adapted to handle weights
  - Grow the set of nodes whose shortest distance has been computed
  - Nodes not in the set will have a “best distance so far”
  - A priority queue will turn out to be useful for efficiency
- An example of a greedy algorithm
  - A series of steps
  - At each one the locally optimal choice is made
Dijkstra’s Algorithm: Idea

- Initially, start node has cost 0 and all other nodes have cost $\infty$
- At each step:
  - Pick closest unknown vertex $v$
  - Add it to the “cloud” of known vertices
  - Update distances for nodes with edges from $v$
- That’s it! (But we need to prove it produces correct answers)
The Algorithm

1. For each node \( v \), set \( v\text{.cost} = \infty \) and \( v\text{.known} = \text{false} \)
2. Set \( \text{source}\text{.cost} = 0 \)
3. While there are unknown nodes in the graph
   a) Select the unknown node \( v \) with lowest cost
   b) Mark \( v \) as known
   c) For each edge \((v,u)\) with weight \( w \),
      \[
      \begin{align*}
      c_1 &= v\text{.cost} + w \quad // \text{cost of best path through } v \text{ to } u \\
      c_2 &= u\text{.cost} \quad // \text{cost of best path to } u \text{ previously known}
      \end{align*}
      \]
      if \( c_1 < c_2 \){ // if the path through \( v \) is better
        \[
        \begin{align*}
        u\text{.cost} &= c_1 \\
        u\text{.path} &= v \quad // \text{for computing actual paths}
        \end{align*}
        \]
Example #1

Order Added to Known Set:

<table>
<thead>
<tr>
<th>vertex</th>
<th>known?</th>
<th>cost</th>
<th>path</th>
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<tbody>
<tr>
<td>A</td>
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Example #1

Order Added to Known Set:

A
Example #1

Order Added to Known Set:
A, C

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Example #1

Order Added to Known Set:
A, C, B

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**Example #1**

Order Added to Known Set:

A, C, B, D

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Example #1

Order Added to Known Set:

A, C, B, D, F

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<tr>
<td>H</td>
<td>Y</td>
<td>7</td>
<td>F</td>
</tr>
</tbody>
</table>

**Order Added to Known Set:**

A, C, B, D, F, H
Example #1

Order Added to Known Set:

A, C, B, D, F, H, G
Example #1

Order Added to Known Set:
A, C, B, D, F, H, G, E

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<tr>
<td>H</td>
<td>Y</td>
<td>7</td>
<td>F</td>
</tr>
</tbody>
</table>
Features

- When a vertex is marked known, the cost of the shortest path to that node is known
  - The path is also known by following back-pointers

- While a vertex is still not known, another shorter path to it might still be found

Note: The “Order Added to Known Set” is not important
  - A detail about how the algorithm works (client doesn’t care)
  - Not used by the algorithm (implementation doesn’t care)
  - It is sorted by path-cost, resolving ties in some way
    - Helps give intuition of why the algorithm works
Interpreting the Results

• Now that we’re done, how do we get the path from, say, A to E?

Order Added to Known Set:
A, C, B, D, F, H, G, E

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</tr>
<tr>
<td>H</td>
<td>Y</td>
<td>7</td>
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</table>
Stopping Short

• How would this have worked differently if we were only interested in:
  – The path from A to G?
  – The path from A to E?

Order Added to Known Set:
A, C, B, D, F, H, G, E
Example #2

Order Added to Known Set:

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Example #2

Order Added to Known Set:
A

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Example #2

Order Added to Known Set:

A, D

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Example #2

Order Added to Known Set:
A, D, C

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Example #2

Order Added to Known Set:
A, D, C, E

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Example #2

Order Added to Known Set:
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Example #2

Order Added to Known Set:
A, D, C, E, B, F
Example #2

Order Added to Known Set:
A, D, C, E, B, F, G

<table>
<thead>
<tr>
<th>vertex</th>
<th>known?</th>
<th>cost</th>
<th>path</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Y</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>Y</td>
<td>3</td>
<td>E</td>
</tr>
<tr>
<td>C</td>
<td>Y</td>
<td>2</td>
<td>A</td>
</tr>
<tr>
<td>D</td>
<td>Y</td>
<td>1</td>
<td>A</td>
</tr>
<tr>
<td>E</td>
<td>Y</td>
<td>2</td>
<td>D</td>
</tr>
<tr>
<td>F</td>
<td>Y</td>
<td>4</td>
<td>C</td>
</tr>
<tr>
<td>G</td>
<td>Y</td>
<td>6</td>
<td>D</td>
</tr>
</tbody>
</table>
Example #3

How will the best-cost-so-far for Y proceed?

Is this expensive?
Example #3

How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, ...

Is this expensive?
Example #3

How will the best-cost-so-far for Y proceed? 90, 81, 72, 63, 54, ...

Is this expensive? No, each edge is processed only once
A Greedy Algorithm

• Dijkstra’s algorithm
  – For single-source shortest paths in a weighted graph (directed or undirected) with no negative-weight edges

• An example of a greedy algorithm:
  – At each step, always does what seems best at that step
    • A locally optimal step, not necessarily globally optimal
  – Once a vertex is known, it is not revisited
    • Turns out to be globally optimal
Where are We?

• Had a problem: Compute shortest paths in a weighted graph with no negative weights

• Learned an algorithm: Dijkstra’s algorithm

• What should we do after learning an algorithm?
  – Prove it is correct
    • Not obvious!
    • We will sketch the key ideas
  – Analyze its efficiency
    • Will do better by using a data structure we learned earlier!
Correctness: Intuition

Rough intuition:

All the “known” vertices have the correct shortest path
  – True initially: shortest path to start node has cost 0
  – If it stays true every time we mark a node “known”, then by induction this holds and eventually everything is “known”

Key fact we need: When we mark a vertex “known” we won’t discover a shorter path later!
  – This holds only because Dijkstra’s algorithm picks the node with the next shortest path-so-far
  – The proof is by contradiction…
Suppose \( v \) is the next node to be marked known (“added to the cloud”)

- The best-known path to \( v \) must have only nodes “in the cloud”
  - Else we would have picked a node closer to the cloud than \( v \)
- Suppose the actual shortest path to \( v \) is different
  - It won’t use only cloud nodes, or we would know about it
  - So it must use non-cloud nodes. Let \( w \) be the first non-cloud node on this path. The part of the path up to \( w \) is already known and must be shorter than the best-known path to \( v \). So \( v \) would not have been picked. Contradiction.
Efficiency, first approach

Use pseudocode to determine asymptotic run-time
  – Notice each edge is processed only once

```python
dijkstra(Graph G, Node start) {
    for each node: x.cost = infinity, x.known = false
    start.cost = 0
    while (not all nodes are known) {
        b = find unknown node with smallest cost
        b.known = true
        for each edge (b, a) in G
            if (!a.known)
                if (b.cost + weight((b, a)) < a.cost) {
                    a.cost = b.cost + weight((b, a))
                    a.path = b
                }
    }
}
```
Efficiency, first approach

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$O(|V|)$
Efficiency, first approach

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                }
    }
}
```

\(\text{O(|V|)}\)  
\(\text{O(|V|^2)}\)
Efficiency, first approach

Use pseudocode to determine asymptotic run-time
  – Notice each edge is processed only once

```plaintext
dijkstra(Graph G, Node start) {
    for each node: x.cost=infinity, x.known=false
    start.cost = 0
    while(not all nodes are known) {
        b = find unknown node with smallest cost
        b.known = true
        for each edge (b,a) in G
            if(!a.known)
                if(b.cost + weight((b,a)) < a.cost){
                    a.cost = b.cost + weight((b,a))
                    a.path = b
                }
    }
}
```

O(|V|)
O(|V|^2)
O(|E|)
**Efficiency, first approach**

Use pseudocode to determine asymptotic run-time

– Notice each edge is processed only once

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- $O(|V|)$
- $O(|V|^2)$
- $O(|E|)$
- $O(|V|^2)$
Improving asymptotic running time

- So far: $O(|V|^2)$

- We had a similar “problem” with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  - We solved it with a queue of zero-degree nodes
  - But here we need the lowest-cost node and costs can change as we process edges

- Solution?
Improving (?) asymptotic running time

• So far: $O(|V|^2)$

• We had a similar “problem” with topological sort being $O(|V|^2)$ due to each iteration looking for the node to process next
  – We solved it with a queue of zero-degree nodes
  – But here we need the lowest-cost node and costs can change as we process edges

• Solution?
  – A priority queue holding all unknown nodes, sorted by cost
  – But must support `decreaseKey` operation
    • Must maintain a reference from each node to its current position in the priority queue
    • Conceptually simple, but can be a pain to code up
Efficiency, second approach

Use pseudocode to determine asymptotic run-time

```python
dijkstra(Graph G, Node start) {
    for each node: x.cost=infinity, x.known=false
    start.cost = 0
    build-heap with all nodes
    while(heap is not empty) {
        b = deleteMin()
        b.known = true
        for each edge (b,a) in G
            if(!a.known)
                if(b.cost + weight((b,a)) < a.cost){
                    decreaseKey(a, "new cost - old cost")
                    a.path = b
                }
    }
}
```
Efficiency, second approach

Use pseudocode to determine asymptotic run-time

dijkstra(Graph G, Node start) {
    for each node: x.cost=\infty, x.known=false
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O(|V|)
Efficiency, second approach

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**Efficiency, second approach**

Use pseudocode to determine asymptotic run-time

\[
dijkstra(G, \text{Node start}) \{
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    \text{while(heap is not empty) } \{
        b = \text{deleteMin()}
        b.\text{known} = \text{true}
        \text{for each edge } (b,a) \text{ in G}
        \quad \text{if(!a.\text{known})}
        \quad \quad \text{if}(b.\text{cost} + \text{weight}((b,a)) < a.\text{cost}) \{
            \text{decreaseKey(a, "new cost - old cost")}
            a.\text{path} = b
        \}
    \}
\]

\[
O(|V|)
\]

\[
O(|V|\log|V|)
\]

\[
O(|E|\log|V|)
\]

\[
O(|V|\log|V| + |E|\log|V|)
\]
Dense vs. sparse again

- First approach: $O(|V|^2)$

- Second approach: $O(|V|\log|V| + |E|\log|V|)$

- So which is better?
  - Sparse: $O(|V|\log|V| + |E|\log|V|)$ (if $|E| > |V|$, then $O(|E|\log|V|)$)
  - Dense: $O(|V|^2)$

- But, remember these are worst-case and asymptotic
  - Priority queue might have slightly worse constant factors
  - On the other hand, for “normal graphs”, we might call `decreaseKey` rarely (or not percolate far), making $|E|\log|V|$ more like $|E|$
Spanning Trees

- A simple problem: Given a connected undirected graph $G=(V,E)$, find a minimal subset of edges such that $G$ is still connected
  - A graph $G_2=(V,E_2)$ such that $G_2$ is connected and removing any edge from $E_2$ makes $G_2$ disconnected
Observations

1. Any solution to this problem is a tree
   - Recall a tree does not need a root; just means acyclic
   - For any cycle, could remove an edge and still be connected

2. Solution not unique unless original graph was already a tree

3. Problem ill-defined if original graph not connected
   - So $|E| \geq |V|-1$

4. A tree with $|V|$ nodes has $|V|-1$ edges
   - So every solution to the spanning tree problem has $|V|-1$ edges
Motivation

A spanning tree connects all the nodes with as few edges as possible

- Example: A “phone tree” so everybody gets the message and no unnecessary calls get made
  - Bad example since would prefer a balanced tree

In most compelling uses, we have a weighted undirected graph and we want a tree of least total cost

- Example: Electrical wiring for a house or clock wires on a chip
- Example: A road network if you cared about asphalt cost rather than travel time

This is the minimum spanning tree problem
  - Will do that next, after intuition from the simpler case
Two Approaches

Different algorithmic approaches to the spanning-tree problem:

1. Do a graph traversal (e.g., depth-first search, but any traversal will do), keeping track of edges that form a tree

2. Iterate through edges; add to output any edge that does not create a cycle