Announcements

• Homework 4 is out
  – Implementing hash tables and hash functions
  – Due Wednesday May 14th at 11pm
  – Allowed to work with a partner

• Midterm next Friday in-class
Midterm, in-class Friday May 9th

- In class, closed notes, closed book.

- Covers everything up to and including hashing.
  - Stacks, queues
  - Induction
  - Asymptotic analysis and Big-Oh
  - Dictionaries, BSTs, AVL Trees
  - Binary heaps and Priority Queues
  - Disjoint sets and Union-Find
  - Hash Tables and Collisions

- Information, sample past exams and solutions posted online.
Graphs

- A graph is a formalism for representing relationships among items
  - Very general definition because very general concept

- A graph is a pair
  \[ G = (V, E) \]
  - A set of vertices, also known as nodes
    \[ V = \{v_1, v_2, \ldots, v_n\} \]
  - A set of edges
    \[ E = \{e_1, e_2, \ldots, e_m\} \]
    - Each edge \( e_i \) is a pair of vertices
      \( (v_j, v_k) \)
    - An edge “connects” the vertices

- Graphs can be directed or undirected
Undirected Graphs

- In **undirected graphs**, edges have no specific direction
  - Edges are always “two-way”

Thus, \((u, v) \in E\) implies \((v, u) \in E\)
  - Only one of these edges needs to be in the set
  - The other is implicit, so normalize how you check for it

- **Degree** of a vertex: number of edges containing that vertex
  - Put another way: the number of adjacent vertices
Directed Graphs

• In directed graphs (sometimes called digraphs), edges have a direction.

  \[ (u, v) \in E \text{ does not imply } (v, u) \in E. \]
  
  • Let \( (u, v) \in E \) mean \( u \rightarrow v \)
  
  • Call \( u \) the source and \( v \) the destination

• In-degree of a vertex: number of in-bound edges, i.e., edges where the vertex is the destination.

• Out-degree of a vertex: number of out-bound edges, i.e., edges where the vertex is the source.
Self-Edges, Connectedness

- A self-edge a.k.a. a loop is an edge of the form \((u, u)\)
  - Depending on the use/algorithm, a graph may have:
    - No self edges
    - Some self edges
    - All self edges (often therefore implicit, but we will be explicit)
- A node can have a degree / in-degree / out-degree of zero
- A graph does not have to be connected
  - Even if every node has non-zero degree
More notation

For a graph $G = (V, E)$:

- $|V|$ is the number of vertices
- $|E|$ is the number of edges
  - Minimum?
  - Maximum for undirected? $|V| (|V|+1)/2 \in O(|V|^2)$
  - Maximum for directed? $|V|^2 \in O(|V|^2)$
    (assuming self-edges allowed, else subtract $|V|$)
- If $(u, v) \in E$
  - Then $v$ is a neighbor of $u$, i.e., $v$ is adjacent to $u$
  - Order matters for directed edges
    - $u$ is not adjacent to $v$ unless $(v, u) \in E$
Examples

Which would use **directed edges**? Which would have **self-edges**?
Which would be **connected**? Which could have **0-degree nodes**?

1. Web pages with links
2. Facebook friends
3. Methods in a program that call each other
4. Road maps (e.g., Google maps)
5. Airline routes
6. Family trees
7. Course pre-requisites
Weighted Graphs

- In a weighed graph, each edge has a weight a.k.a. cost
  - Typically numeric (most examples use ints)
  - Orthogonal to whether graph is directed
  - Some graphs allow negative weights; many do not

![Diagram of weighted graph with cities and weights]
Examples

What, if anything, might weights represent for each of these? Do negative weights make sense?

- Web pages with links
- Facebook friends
- Methods in a program that call each other
- Road maps (e.g., Google maps)
- Airline routes
- Family trees
- Course pre-requisites
Paths and Cycles

- A path is a list of vertices \([v_0, v_1, \ldots, v_n]\) such that \((v_i, v_{i+1}) \in E\) for all \(0 \leq i < n\). Say “a path from \(v_0\) to \(v_n\)”

- A cycle is a path that begins and ends at the same node \((v_0 = = v_n)\)

Example: [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]
Path Length and Cost

- Path length: Number of edges in a path
- Path cost: Sum of weights of edges in a path

Example where
P = [Seattle, Salt Lake City, Chicago, Dallas, San Francisco, Seattle]

length(P) = 5
cost(P) = 11.5
Simple Paths and Cycles

- A simple path repeats no vertices, except the first might be the last
  [Seattle, Salt Lake City, San Francisco, Dallas]
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]

- Recall, a cycle is a path that ends where it begins
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
  [Seattle, Salt Lake City, Seattle, Dallas, Seattle]

- A simple cycle is a cycle and a simple path
  [Seattle, Salt Lake City, San Francisco, Dallas, Seattle]
Paths and Cycles in Directed Graphs

Example:

Is there a path from A to D?  No

Does the graph contain any cycles?  No
**Undirected-Graph Connectivity**

- An undirected graph is **connected** if for all pairs of vertices $u, v$, there exists a *path* from $u$ to $v$.

![Connected graph](image)

**Disconnected graph**

- An undirected graph is **complete**, a.k.a. **fully connected** if for all pairs of vertices $u, v$, there exists an *edge* from $u$ to $v$, and possibly self edges.

![Complete graph](image)
Directed-Graph Connectivity

- A directed graph is **strongly connected** if there is a path from every vertex to every other vertex.

- A directed graph is **weakly connected** if there is a path from every vertex to every other vertex *ignoring direction of edges*.

- A **complete** a.k.a. **fully connected** directed graph has an edge from every vertex to every other vertex *plus self edges*.
Trees as Graphs

When talking about graphs, we say a tree is a graph that is:

- Undirected
- Acyclic
- Connected

So all trees are graphs, but not all graphs are trees
Rooted Trees

- We are more accustomed to rooted trees where:
  - We identify a unique root
  - We think of edges as directed: parent to children

- Given a tree, picking a root gives a unique rooted tree
  - The tree is just drawn differently

```
A      A
|      |   redrawn
B ----|----- B
C    D  C
F ----E  F
G    H  G
```
Rooted Trees

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Directed Acyclic Graphs (DAGs)

- A **DAG** is a directed graph with no (directed) cycles
  - Every rooted directed tree is a DAG
  - But not every DAG is a rooted directed tree

- Every DAG is a directed graph
- But not every directed graph is a DAG
Examples

Which of our directed-graph examples do you expect to be a DAG?

- Web pages with links
- Methods in a program that call each other
- Airline routes
- Family trees
- Course pre-requisites
Density / Sparsity

- Recall: In an undirected graph, $0 \leq |E| < |V|^2$
- Recall: In a directed graph: $0 \leq |E| \leq |V|^2$
- So for any graph, $O(|E|+|V|^2)$ is $O(|V|^2)$
- Another fact: If an undirected graph is connected, then $|V|-1 \leq |E|$
- Because $|E|$ is often much smaller than its maximum size, we do not always approximate $|E|$ as $O(|V|^2)$
  - This is a correct bound, it just is often not tight
  - If it is tight, i.e., $|E|$ is $\Theta(|V|^2)$ we say the graph is dense
    - More sloppily, dense means “lots of edges”
  - If $|E|$ is $O(|V|)$ we say the graph is sparse
    - More sloppily, sparse means “most possible edges missing”
What is the Data Structure?

• So graphs are really useful for lots of data and questions
  – For example, “what’s the lowest-cost path from x to y”

• But we need a data structure that represents graphs

• The “best one” can depend on:
  – Properties of the graph (e.g., dense versus sparse)
  – The common queries (e.g., “is \((u, v)\) an edge?” versus “what are the neighbors of node \(u\)?”)

• So we’ll discuss the two standard graph representations
  – Adjacency Matrix and Adjacency List
  – Different trade-offs, particularly time versus space
Adjacency Matrix

- Assign each node a number from 0 to $|V| - 1$
- A $|V| \times |V|$ matrix (i.e., 2-D array) of Booleans (or 1 vs. 0)
  - If $M$ is the matrix, then $M[u][v]$ being true means there is an edge from $u$ to $v$

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### Adjacency Matrix Properties

- **Running time to:**
  - Get a vertex’s out-edges: $O(|V|)$
  - Get a vertex’s in-edges: $O(|V|)$
  - Decide if some edge exists: $O(1)$
  - Insert an edge: $O(1)$
  - Delete an edge: $O(1)$

- **Space requirements:**
  - $|V|^2$ bits

- **Best for sparse or dense graphs?**
  - Best for dense graphs

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Adjacency Matrix Properties

• How will the adjacency matrix vary for an undirected graph?
  – Undirected will be symmetric around the diagonal

• How can we adapt the representation for weighted graphs?
  – Instead of a Boolean, store a number in each cell
  – Need some value to represent ‘not an edge’
    • In some situations, 0 or -1 works
**Adjacency List**

- Assign each node a number from 0 to $|V| - 1$
- An array of length $|V|$ in which each entry stores a list of all adjacent vertices (e.g., linked list)
Adjacency List Properties

- Running time to:
  - Get all of a vertex’s out-edges: $O(d)$ where $d$ is out-degree of vertex
  - Get all of a vertex’s in-edges: $O(|E|)$ (but could keep a second adjacency list for this!)
  - Decide if some edge exists: $O(d)$ where $d$ is out-degree of source
  - Insert an edge: $O(1)$ (unless you need to check if it’s there)
  - Delete an edge: $O(d)$ where $d$ is out-degree of source

- Space requirements: $O(|V|+|E|)$

  - Good for sparse graphs
Next…

Okay, we can represent graphs

Next lecture we’ll implement some useful and non-trivial algorithms

- **Topological sort**: Given a DAG, order all the vertices so that every vertex comes before all of its neighbors

- **Shortest paths**: Find the shortest or lowest-cost path from x to y
  - Related: Determine if there even is such a path