Minimum Spanning Trees

CSE 373
Data Structures & Algorithms
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Today’s Outline

• Announcements:
  › HW 5 due Friday, May 31.

• Today’s Topics:
  › Weiss 9.5, 9.6

Recall Spanning Tree

• Given (connected) graph G(V,E), a spanning tree T(V',E'):
  › Spans the graph (V' = V)
  › Forms a tree (no cycle);
  › E' has |V| - 1 edges

Minimum Spanning Tree

• Edges are weighted: find minimum cost spanning tree
• Applications
  › Find cheapest way to wire your house
  › Find minimum cost to send a message on the Internet

Strategy for Minimum Spanning Tree

• For any spanning tree T, inserting an edge e_{new} not in T creates a cycle
• But
  › Removing any edge e_{old} from the cycle gives back a spanning tree
  › If e_{new} has a lower cost than e_{old} we have progressed!

Strategy

• Strategy for construction:
  › Add an edge of minimum cost that does not create a cycle (greedy algorithm)
  › Repeat |V| - 1 times
  › Correct since if we could replace an edge with one of lower cost, the algorithm would have picked it up
Two Algorithms

- **Prim**: (build tree incrementally)
  - Pick lower cost edge connected to known (incomplete) spanning tree that does not create a cycle and expand to include it in the tree
- **Kruskal**: (build forest that will finish as a tree)
  - Pick lower cost edge not yet in a tree that does not create a cycle and expand to include it somewhere in the forest

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**Prim's algorithm**

Starting from empty T, choose a vertex at random and initialize V = {1}, E' = {}.


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Choose the vertex u not in V such that edge weight from u to a vertex in V is minimal (greedy!)
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V = {1, 3} E' = {(1, 3)}

Repeat until all vertices have been chosen:

Choose the vertex u not in V such that edge weight from v to a vertex in V is minimal (greedy!)

V = {1, 3, 4} E' = {(1, 3), (3, 4)}

...

V = {1, 3, 4, 5, 2, 6}
E' = {(1, 3), (3, 4), (4, 5), (5, 2), (2, 6)}

Final Cost: 1 + 3 + 4 + 1 + 1 = 10

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**Prim's algorithm**

Repeat until all vertices have been chosen.

V = {1, 3, 4, 5, 2, 6}
E' = {(1, 3), (3, 4), (4, 5), (5, 2), (2, 6)}

Final Cost: 1 + 3 + 4 + 1 + 1 = 10
Prim's algorithm

Repeat until all vertices have been chosen.

2. Choose a vertex not in tree with minimal cost edge to add.

Prim's algorithm

Repeat until all vertices have been chosen.

3. Choose a vertex not in tree with minimal cost edge to add.

Prim's algorithm

Repeat until all vertices have been chosen.

4. Choose a vertex not in tree with minimal cost edge to add.

Prim's algorithm

Repeat until all vertices have been chosen.

5. Choose a vertex not in tree with minimal cost edge to add.

Prim's algorithm

Repeat until all vertices have been chosen.

6. Choose a vertex not in tree with minimal cost edge to add.

Prim's Algorithm Implementation

- Assume adjacency list representation
- Initialize connection cost of each node to “inf” and “unmark” them
- Choose one node, say v and set cost[v] = 0 and prev[v] = 0
- While there are unmarked nodes
  - Select the unmarked node u with minimum cost; mark it
  - For each unmarked node w adjacent to u
    - if cost(u, w) < cost(w) then {
      - cost(w) := cost(u, w)
      - prev[w] = u
    }
- Looks a lot like Dijkstra’s algorithm!
Prim's algorithm

Repeat until all vertices have been chosen.

1. Choose an initial vertex and put in tree.

2. Choose min cost vertex not in tree and put in tree.
   Update the costs of its neighbors.

3. Choose min cost vertex not in tree and put in tree.
   Update the costs of its neighbors.

4. Choose min cost vertex not in tree and put in tree.
   Update the costs of its neighbors.

Prim's algorithm Analysis

• Like Dijkstra’s algorithm
• If the “Select the unmarked node u with minimum cost” is done with binary heap then $O((n+m)\log n)$
Kruskal’s Algorithm

- Select edges in order of increasing cost
- Accept an edge to expand tree or forest only if it does not cause a cycle
- Implementation using adjacency list, priority queues and disjoint sets

Detecting Cycles

- If the edge to be added \((u,v)\) is such that vertices \(u\) and \(v\) belong to the same tree, then by adding \((u,v)\) you would form a cycle
  - Therefore to check, \(\text{Find}(u)\) and \(\text{Find}(v)\). If they are the same discard \((u,v)\)
  - If they are different \(\text{Union}(\text{Find}(u), \text{Find}(v))\)

Properties of trees in K’s algorithm

- Vertices in different trees are disjoint
  - True at initialization and Union won’t modify the fact for remaining trees
- Trees form equivalent classes under the relation “is connected to”
  - \(u\) connected to \(u\) (reflexivity)
  - \(u\) connected to \(v\) implies \(v\) connected to \(u\) (symmetry)
  - \(u\) connected to \(v\) and \(v\) connected to \(w\) implies a path from \(u\) to \(w\) so \(u\) connected to \(w\) (transitivity)

K’s Algorithm Data Structures

- Adjacency list for the graph
  - To perform the initialization of the data structures below
- Disjoint Set ADT’s for the trees (recall Up tree implementation of Union-Find)
- Binary heap for edges

Example
Initialization
Initially, Forest of 6 trees
\(F= \{(1),(2),(3),(4),(5),(6)\}\)

Edges in a heap (not shown)

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{array}
\]

Step 1
Select edge with lowest cost (2,5)
Find(2) = 2, Find (5) = 5
Union(2,5)
\(F= \{(1),(2,5),(3),(4),(6)\}\)
1 edge accepted

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{array}
\]

Step 2
Select edge with lowest cost (2,6)
Find(2) = 2, Find (6) = 6
Union(2,6)
\(F= \{(1),(2,5,6),(3),(4)\}\)
2 edges accepted

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{array}
\]

Step 3
Select edge with lowest cost (1,3)
Find(1) = 1, Find (3) = 3
Union(1,3)
\(F= \{(1,3),(2,5,6),(4)\}\)
3 edges accepted

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{array}
\]

Step 4
Select edge with lowest cost (5,6)
Find(5) = 2, Find (6) = 2
Do nothing
\(F= \{(1,3),(2,5,6),(4)\}\)
3 edges accepted

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{array}
\]

Step 5
Select edge with lowest cost (3,4)
Find(3) = 1, Find (4) = 4
Union(1,4)
\(F= \{(1,3,4),(2,5,6)\}\)
4 edges accepted

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
\end{array}
\]
Step 6

Select edge with lowest cost (4,5)
Find(4) = 1, Find (5) = 2
Union(1,2)
F= {{1,3,4,2,5,6}}
5 edges accepted : end
Total cost = 10
Although there is a unique spanning tree in this example, this is not generally the case

Kruskal’s Algorithm Analysis

- Initialize forest O(n)
- Initialize heap O(m), m = |E|
- Loop performed m times
  - In the loop one Deletemin O(logm)
  - Two Find, each O(logn)
  - One Union (at most) O(1)
- So worst case O(mlogm) = O(mlogn)

Time Complexity Summary

- Recall that m = |E| = O(V^2) = O(n^2)
- Prim’s runs in O((n+m) log n)
- Kruskal’s runs in O(m log m)
- In practice, Kruskal has a tendency to run faster since graphs might not be dense and not all edges need to be looked at in the Deletemin operations