CSE373: Data Structures & Algorithms
Lecture 9: Disjoint Sets & Union-Find

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The plan

- What are *disjoint sets*
  - And how are they "the same thing" as *equivalence relations*
- The union-find ADT for disjoint sets
- Applications of union-find

Next lecture:

- Basic implementation of the ADT with "up trees"
- Optimizations that make the implementation much faster

Disjoint sets

- A set is a collection of elements (no-repeats)
- Two sets are disjoint if they have no elements in common
  - $\mathbf{S_1 \cap S_2 = \varnothing}$
- Example: {a, e, c} and {d, b} are disjoint
- Example: {x, y, z} and {t, u, x} are not disjoint

Partitions

A partition *P* of a set *S* is a set of sets {*S1*,*S2*,…,*Sn*} such that every element of *S* is in **exactly one** *Si*

Put another way:
- $S_1 \cup S_2 \cup \ldots \cup S_k = S$
- $i \neq j$ implies $S_i \cap S_j = \varnothing$ (sets are disjoint with each other)

Example:
- Let *S* be {a,b,c,d,e}
- One partition: {a}, {d,e}, {b,c}
- Another partition: {a,b,c}, ∅, {d}, {e}
- A third: {a,b,c,d,e}
- Not a partition: {a,b,d}, {c,d,e}
- Not a partition of *S*: {a,b}, {e,c}

Binary relations

- *S x S* is the set of all pairs of elements of *S*
  - Example: If *S* = {a,b,c}
    then *S x S* = {(a,a),(a,b),(a,c),(b,a),(b,b),(b,c), (c,a),(c,b),(c,c)}
- A binary relation *R* on a set *S* is any subset of *S x S*
  - Write *R*(x,y) to mean (x,y) is "in the relation"
  - (Unary, ternary, quaternary, … relations defined similarly)
- Examples for *S* = people-in-this-room
  - Sitting-next-to-each-other relation
  - First-sitting-right-of-second relation
  - Went-to-same-high-school relation
  - Same-gender-relation
  - First-is-younger-than-second relation

Properties of binary relations

- A binary relation *R* over set *S* is reflexive means
  R(a,a) for *all* a in *S*
- A binary relation *R* over set *S* is symmetric means
  R(a,b) if and only if *R*(b,a) for *all* a,b in *S*
- A binary relation *R* over set *S* is transitive means
  If *R*(a,b) and *R*(b,c) then *R*(a,c) for *all* a,b,c in *S*
- Examples for *S* = people-in-this-room
  - Sitting-next-to-each-other relation
  - First-sitting-right-of-second relation
  - Went-to-same-high-school relation
  - Same-gender-relation
  - First-is-younger-than-second relation

Equivalence relations

- A binary relation *R* is an equivalence relation if *R* is reflexive, symmetric, and transitive
- Examples
  - Same gender
  - Connected roads in the world
  - *Graduated* from same high school?
  - …

Punch-line

- Every partition induces an *equivalence relation*
- Every equivalence relation *induces* a partition
- Suppose *P*={*S1*,*S2*,…,*Sn*} be a partition
  - Define *R*(x,y) to mean x and y are in the same *Si*
    - *R* is an equivalence relation
- Suppose *R* is an equivalence relation over *S*
  - Consider a set of sets S1,S2,…,Sn where
    (1) x and y are in the same Si if and only if R(x,y)
    (2) Every x is in some Si
    - This set of sets is a partition

Example

- Let *S* be {a,b,c,d,e}
- One partition: {a,b,c}, {d}, {e}
- The corresponding equivalence relation:
  (a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (d,d), (e,e)

The plan

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  - And how are they "the same thing" as *equivalence relations*
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- Applications of union-find

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The operations

- Given an unchanging set *S*, **`create`** an initial partition of a set
  - Typically each item in its own subset: {a}, {b}, {c}, …
  - Give each subset a "name" by choosing a *representative element*
- Operation **`find`** takes an element of *S* and returns the representative element of the subset it is in
- Operation **`union`** takes two subsets and (permanently) makes one larger subset
  - A different partition with one fewer set
  - Affects result of subsequent **`find`** operations
  - Choice of representative element up to implementation

Example

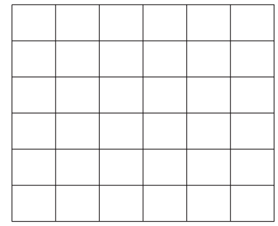
- Let *S* = {1,2,3,4,5,6,7,8,9}
- Let initial partition be (will highlight representative elements red)
  {1}, {2}, {3}, {4}, {5}, {6}, {7}, {8}, {9}
- **`union`**(2,5):
  {1}, {2, 5}, {3}, {4}, {6}, {7}, {8}, {9}
- **`find`**(4) = 4, **`find`**(2) = 2, **`find`**(5) = 2
- **`union`**(4,6), **`union`**(2,7)
  {1}, {2, 5, 7}, {3}, {4, 6}, {8}, {9}
- **`find`**(4) = 6, **`find`**(2) = 2, **`find`**(5) = 2
- **`union`**(2,6)
  {1}, {2, 4, 5, 6, 7}, {3}, {8}, {9}

No other operations

- All that can "happen" is sets get unioned
  - No "un-union" or "create new set" or …
- As always: trade-offs – implementations will exploit this small ADT
- Surprisingly useful ADT: list of applications after one example surprising one
  - But not as common as dictionaries or priority queues

Example application: maze-building

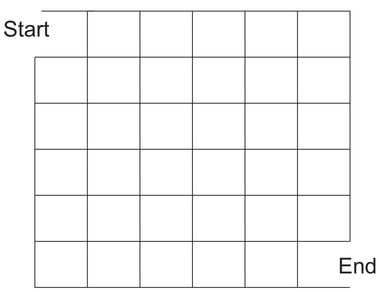
- Build a random maze by erasing edges



  - Possible to get from anywhere to anywhere
    - Including "start" to "finish"
  - No loops possible without backtracking
    - After a "bad turn" have to "undo"

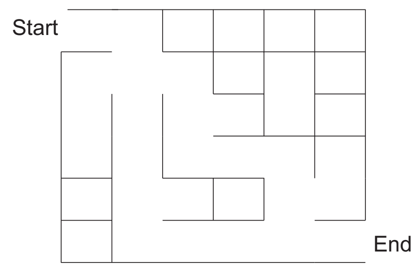
Maze building

Pick start edge and end edge



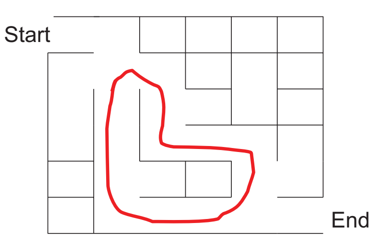
Repeatedly pick random edges to delete

One approach: just keep deleting random edges until you can get from start to finish



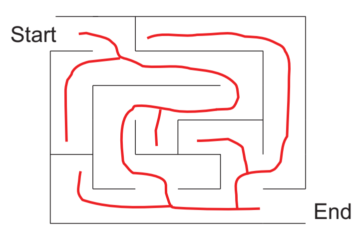
Problems with this approach

1. How can you tell when there is a path from start to finish?
   - We do not really have an algorithm yet
2. We have *cycles*, which a "good" maze avoids
   - Want one solution and no cycles



Revised approach

- Consider edges in random order
- But only delete them if they introduce no cycles (how? TBD)
- When done, will have one way to get from any place to any other place (assuming no backtracking)



- Notice the funny-looking *tree* in red

Cells and edges

- Let's number each cell
  - 36 total for 6 x 6
- An (internal) edge (x,y) is the line between cells x and y
  - 60 total for 6x6: (1,2), (2,3), …, (1,7), (2,8), …

Start

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

The trick

- Partition the cells into **disjoint sets**: "are they connected"
  - Initially every cell is in its own subset
- If an edge would connect two different subsets:
  - then remove the edge and **union** the subsets
  - else leave the edge because removing it makes a cycle

Start

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

Start

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

The algorithm

- P = **disjoint sets** of connected cells, initially each cell in its own 1-element set
- E = **set** of edges not yet processed, initially all (internal) edges
- M = **set** of edges kept in maze (initially empty)

while P has more than one set {
- Pick a random edge (x,y) to remove from E
- u = **find**(x)
- v = **find**(y)
- if u==v

  then add (x,y) to M // same subset, do not create cycle

  else **union**(u,v) // do not put edge in M, connect subsets

}

Add remaining members of E to M, then output M as the maze

Example step

Pick (8,14)

Start

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

Example step

P
{1,2,7,8,9,13,19}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{14,20,26,27}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

Find(8) = 7
Find(14) = 20

Union(7,20)

P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

Add edge to M step

Pick (19,20)

Start

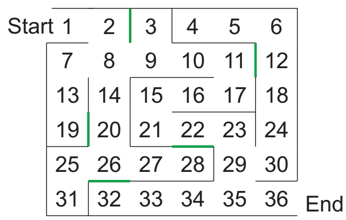
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

End

P
{1,2,7,8,9,13,19,14,20,26,27}
{3}
{4}
{5}
{6}
{10}
{11,17}
{12}
{15,16,21}
{18}
{25}
{28}
{31}
{22,23,24,29,30,32
33,34,35,36}

At the end

- Stop when P has one set
- Suppose green edges are already in M and black edges were not yet picked
  - Add all black edges to M



P
{1,2,3,4,5,6,7,… 36}

Other applications

- Maze-building is:
  - Cute
  - Homework 4 ☺
  - A surprising use of the union-find ADT
- Many other uses (which is why an ADT taught in CSE373):
  - Road/network/graph connectivity (will see this again)
    - "connected components" e.g., in social network
  - Partition an image by connected-pixels-of-similar-color
  - Type inference in programming languages
- Not as common as dictionaries, queues, and stacks, but valuable because implementations are very fast, so when applicable can provide big improvements