CSE373: Data Structures & Algorithms
Lecture 12: Hash Collisions

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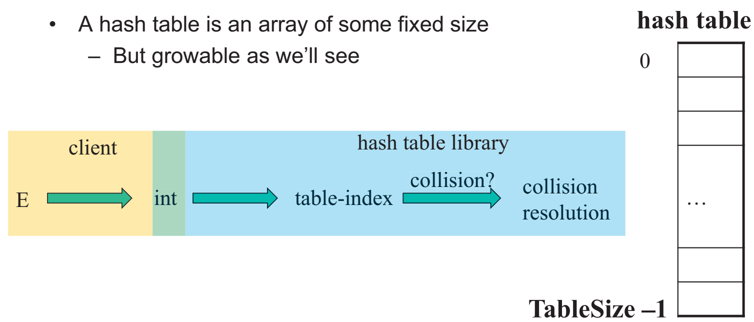
Hash Tables: Review

- Aim for constant-time (i.e., *O*(1)) **`find`**, **`insert`**, and **`delete`**
  - "On average" under some reasonable assumptions
- A hash table is an array of some fixed size
  - But growable as we'll see



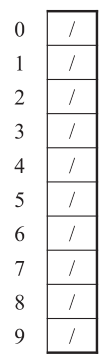
Collision resolution

Collision:
When two keys map to the same location in the hash table

We try to avoid it, but number-of-keys exceeds table size

So hash tables should support collision resolution
- Ideas?

Separate Chaining

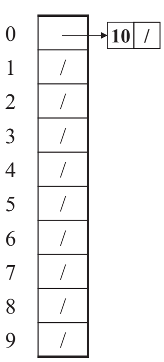


Chaining:
All keys that map to the same table location are kept in a list (a.k.a. a "chain" or "bucket")

As easy as it sounds

Example:
insert 10, 22, 107, 12, 42
with mod hashing
and **`TableSize`** = 10

Separate Chaining



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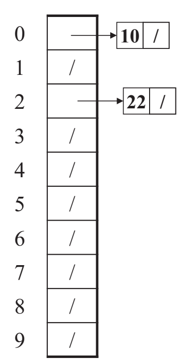

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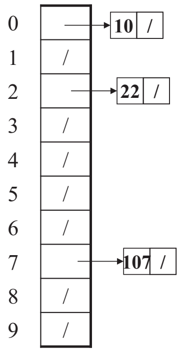

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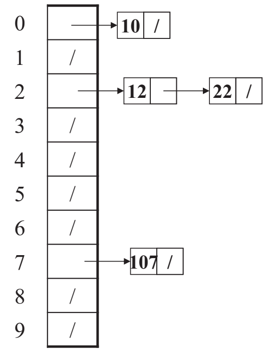

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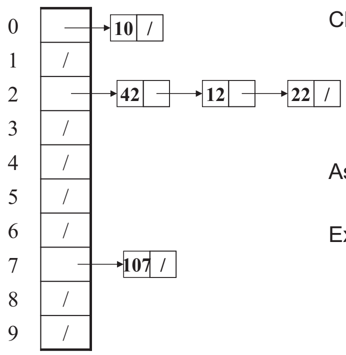

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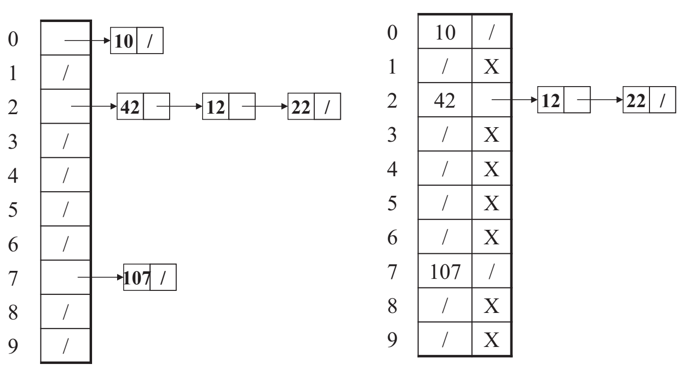
Thoughts on chaining

- Worst-case time for **`find`**?
  - Linear
  - But only with really bad luck or bad hash function
  - So not worth avoiding (e.g., with balanced trees at each bucket)
- Beyond asymptotic complexity, some "data-structure engineering" may be warranted
  - Linked list vs. array vs. chunked list (lists should be short!)
  - Move-to-front
  - Maybe leave room for 1 element (or 2?) in the table itself, to optimize constant factors for the common case
    - A time-space trade-off…

Time vs. space (constant factors only here)



More rigorous chaining analysis

Definition: The load factor, λ, of a hash table is

$$\lambda = \frac{N}{\text{TableSize}}$$ **← number of elements**

Under chaining, the average number of elements per bucket is ____

So if some inserts are followed by *random* finds, then on average:
- Each unsuccessful **`find`** compares against ____ items
- Each successful **`find`** compares against _____ items

More rigorous chaining analysis

Definition: The load factor, λ, of a hash table is

$$\lambda = \frac{N}{\text{TableSize}} \quad \leftarrow \textbf{number of elements}$$

Under chaining, the average number of elements per bucket is λ

So if some inserts are followed by *random* finds, then on average:

- Each unsuccessful **`find`** compares against λ items
- Each successful **`find`** compares against $\lambda/2$ items

So we like to keep λ fairly low (e.g., 1 or 1.5 or 2) for chaining

Alternative: Use empty space in the table

- Another simple idea: If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full…
- Example: insert 38, 19, 8, 109, 10

0	/
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	/

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1	/
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4	/
5	/
6	/
7	/
8	38
9	19

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0	8
1	/
2	/
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0	8
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3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open addressing

This is *one example* of open addressing

In general, open addressing means resolving collisions by trying a sequence of other positions in the table

Trying the next spot is called probing

- We just did linear probing
  - i^{th} probe was `(h(key) + i) % TableSize`
- In general have some probe function `f` and use `h(key) + f(i) % TableSize`

Open addressing does poorly with high load factor λ

- So want larger tables
- Too many probes means no more $O(1)$

Terminology

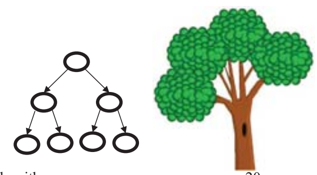
We and the book use the terms

- "chaining" or "separate chaining"
- "open addressing"

Very confusingly,

- "open hashing" is a synonym for "chaining"
- "closed hashing" is a synonym for "open addressing"

(If it makes you feel any better,
most trees in CS grow upside-down ☺)



Other operations

`insert` finds an open table position using a probe function

What about `find`?

- Must use same probe function to "retrace the trail" for the data
- Unsuccessful search when reach empty position

What about `delete`?

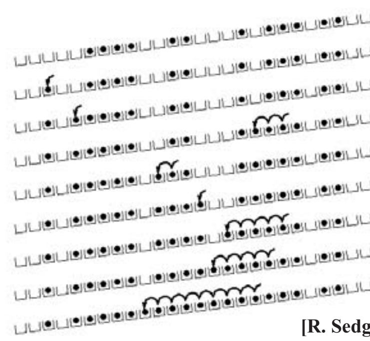
- ***Must*** use "lazy" deletion. Why?
  - Marker indicates "no data here, but don't stop probing"
- Note: `delete` with chaining is plain-old list-remove

(Primary) Clustering

It turns out linear probing is a *bad idea*, even though the probe function is quick to compute (which is a good thing)

Tends to produce *clusters*, which lead to long probing sequences

- Called primary clustering
- Saw this starting in our example



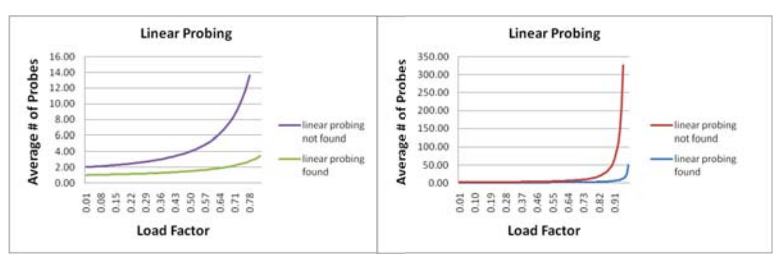
[R. Sedgewick]

Analysis of Linear Probing

- Trivial fact: For any $\lambda < 1$, linear probing will find an empty slot
  - It is "safe" in this sense: no infinite loop unless table is full
- Non-trivial facts we won't prove:
  Average # of probes given λ (in the limit as `TableSize` $\rightarrow \infty$)
  - Unsuccessful search: $\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$
  - Successful search: $\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$
- This is pretty bad: need to leave sufficient empty space in the table to get decent performance (see chart)

In a chart

- Linear-probing performance degrades rapidly as table gets full
  - (Formula assumes "large table" but point remains)



- By comparison, chaining performance is linear in λ and has no trouble with $\lambda > 1$

Quadratic probing

- We can avoid primary clustering by changing the probe function
  `(h(key) + f(i)) % TableSize`
- A common technique is quadratic probing:
  `f(i) = i`^{2}
  - So probe sequence is:
    - 0th probe: `h(key) % TableSize`
    - 1st probe: `(h(key) + 1) % TableSize`
    - 2nd probe: `(h(key) + 4) % TableSize`
    - 3rd probe: `(h(key) + 9) % TableSize`
    - …
    - ith probe: `(h(key) + i`^{2}`) % TableSize`
- Intuition: Probes quickly "leave the neighborhood"

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

TableSize=10
Insert:
89
18
49
58
79

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	89

TableSize=10
Insert:
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Quadratic Probing Example

0	49
1	
2	58
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4	
5	
6	
7	
8	18
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TableSize=10
Insert:
89
18
49
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79

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	

TableSize = 7

Insert:
76 (76 % 7 = 6)
40 (40 % 7 = 5)
48 (48 % 7 = 6)
5 (5 % 7 = 5)
55 (55 % 7 = 6)
47 (47 % 7 = 5)

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	76

TableSize = 7

Insert:
76 (76 % 7 = 6)
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Insert:
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Insert:
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Another Quadratic Probing Example

0	48
1	
2	5
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TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

Another Quadratic Probing Example

0	48
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TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

Doh!: For all *n*, `((n*n) +5) % 7 is 0, 2, 5, or 6`

- Excel shows takes "at least" 50 probes and a pattern
- Proof uses induction and $(n^2+5) \% 7 = ((n-7)^2+5) \% 7$
  - In fact, for all *c* and *k*, $(n^2+c) \% k = ((n-k)^2+c) \% k$

From Bad News to Good News

- Bad news:
  - Quadratic probing can cycle through the same full indices, never terminating despite table not being full
- Good news:
  - If **`TableSize`** is *prime* and λ < ½, then quadratic probing will find an empty slot in at most **`TableSize/2`** probes
  - So: If you keep λ < ½ and **`TableSize`** is *prime*, no need to detect cycles
  - Optional: Proof is posted in **`lecture12.txt`**
    - Also, slightly less detailed proof in textbook
    - Key fact: For prime **`T`** and **`0 < i,j < T/2`** where **`i ≠ j`**, $(k + i^2) \% T \neq (k + j^2) \% T$ (i.e., no index repeat)

Clustering reconsidered

- Quadratic probing does not suffer from primary clustering: no problem with keys initially hashing to the same neighborhood
- But it's no help if keys initially hash to the same index
  - Called secondary clustering
- Can avoid secondary clustering with a probe function that depends on the key: double hashing…

Double hashing

Idea:

- Given two good hash functions *h* and *g*, it is very unlikely that for some *key*, **`h(key) == g(key)`**
- So make the probe function **`f(i) = i*g(key)`**

Probe sequence:

- 0^{th} probe: **`h(key) % TableSize`**
- 1^{st} probe: **`(h(key) + g(key)) % TableSize`**
- 2^{nd} probe: **`(h(key) + 2*g(key)) % TableSize`**
- 3^{rd} probe: **`(h(key) + 3*g(key)) % TableSize`**
- …
- i^{th} probe: **`(h(key) + i*g(key)) % TableSize`**

Detail: Make sure **`g(key)`** cannot be **`0`**

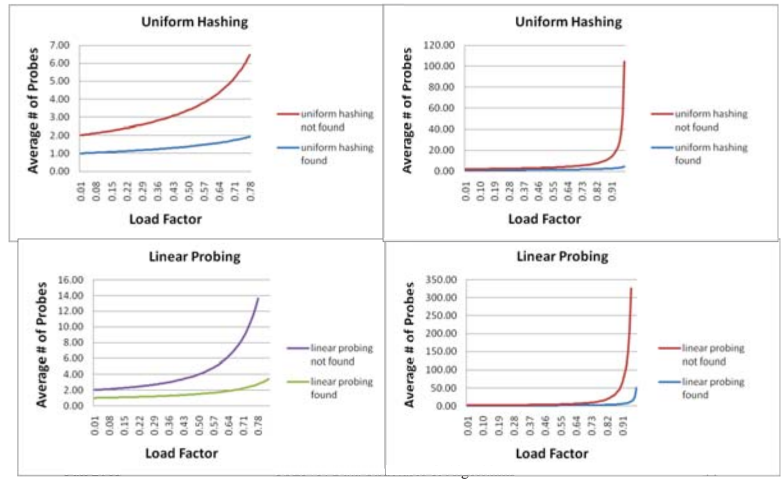
Double-hashing analysis

- Intuition: Because each probe is "jumping" by **`g(key)`** each time, we "leave the neighborhood" *and* "go different places from other initial collisions"
- But we could still have a problem like in quadratic probing where we are not "safe" (infinite loop despite room in table)
  - It is known that this cannot happen in at least one case:
    - **`h(key) = key % p`**
    - **`g(key) = q – (key % q)`**
    - **`2 < q < p`**
    - **`p`** and **`q`** are prime

More double-hashing facts

- Assume "uniform hashing"
  - Means probability of `g(key1) % p == g(key2) % p` is `1/p`
- Non-trivial facts we won't prove:
  Average # of probes given λ (in the limit as `TableSize` $\rightarrow\infty$)
  - Unsuccessful search (intuitive): $$\frac{1}{1-\lambda}$$
  - Successful search (less intuitive): $$\frac{1}{\lambda}\log_e\left(\frac{1}{1-\lambda}\right)$$
- Bottom line: unsuccessful bad (but not as bad as linear probing), but successful is not nearly as bad

Charts



Rehashing

- As with array-based stacks/queues/lists, if table gets too full, create a bigger table and copy everything
- With chaining, we get to decide what "too full" means
  - Keep load factor reasonable (e.g., < 1)?
  - Consider average or max size of non-empty chains?
- For open addressing, half-full is a good rule of thumb
- New table size
  - Twice-as-big is a good idea, except, uhm, that won't be prime!
  - So go *about* twice-as-big
  - Can have a list of prime numbers in your code since you won't grow more than 20-30 times