The AVL Balance Condition

Left and right subtrees of every node have equal heights differing by at most 1

Define: \( \text{balance}(x) = \text{height}(x, \text{left}) - \text{height}(x, \text{right}) \)

AVL property: \(-1 \leq \text{balance}(x) \leq 1, \text{ for every node } x\)

- Ensures small depth
- Will prove this by showing that an AVL tree of height \( h \) must have a lot of (i.e. \( \Theta(2^h) \)) nodes
- Easy to maintain
- Using single and double rotations

The AVL Tree Data Structure

Structural properties
1. Binary tree property (0, 1, or 2 children)
2. Heights of left and right subtrees of every node differ by at most 1

Result:
- Worst case depth of any node is: \( O(\log n) \)

Ordering property
- Same as for BST
Is this an AVL Tree?

NULLs have height -1

Circle One:
AVL
Not AVL
AVL
Not AVL

Student Activity: If not AVL, put a box around nodes where AVL property is violated.

Proving Shallowness Bound

Let \( S(h) \) be the min # of nodes in an AVL tree of height \( h \)

Claim: \( S(h) = S(h-1) + S(h-2) + 1 \)

Solution of recurrence: \( S(h) = \Theta(2^h) \) (like Fibonacci numbers)

An AVL Tree
AVL trees: find, insert

- **AVL find:**
  - same as BST find.

- **AVL insert:**
  - same as BST insert, except may need to “fix” the AVL tree after inserting new value.

AVL Insert: detect & fix imbalances

1. Insert the new node just as you would in a BST (as a new leaf).
2. For each node on the path from the inserted node up to the root, the insertion may (or may not) have changed the node’s height.
3. So after recursive insertion in a subtree, check for height imbalance at each of these nodes and perform a rotation to restore balance at that node if needed.

All the action is in defining the correct rotations to restore balance.

Fact that makes it a bit easier:
- There must be a deepest node that is imbalanced after the insert (all descendants still balanced).
- After rebalancing this deepest node, every node is balanced.
- So at most one node needs to be rebalanced.

AVL tree insert

Let x be the node where an imbalance occurs.

Four cases to consider. The insertion is in the:
1. left subtree of the left child of x.
2. right subtree of the left child of x.
3. left subtree of the right child of x.
4. right subtree of the right child of x.

Idea: Cases 1 & 4 are solved by a *single rotation*.
Cases 2 & 3 are solved by a *double rotation*.

AVL Insert: detect & fix imbalances

1. Insert(6)
2. Insert(3)
3. Insert(1)

Bad Case #1

1. Insert(6)
2. Insert(3)
3. Insert(1)
Bad Case #1: Example

Insert(6)
Insert(3)
Insert(1)

Third insertion violates balance property
- happens to be at the root

What is the only way to fix this?

Fix: Apply “Single Rotation”

Single rotation: The basic operation we’ll use to rebalance
- Move child of unbalanced node into parent position
- Parent becomes the “other” child (always okay in a BST!)
- Other subtrees move in only way BST allows (next slide)

AVL Property violated at this node ("x")

Single Rotation:
1. Rotate between “x” and child

Generalized left-left case

- Node \( a \) imbalanced due to insertion somewhere in left-left grandchild increasing height of left subtree.
  - 1 of 4 possible imbalance causes (other three coming)

- First we did the insertion, which makes \( a \) imbalanced:

Before insertion – balanced

After insertion – unbalanced

Generalized left-left case (cont.)

- So we rotate at \( a \), using BST facts: \( X < b < Y < a < Z \)
- A single rotation to the right restores balance at the node
  - To same height as before insertion (so ancestors now balanced)

\[ \text{After insertion – unbalanced!} \]

\[ \text{After single rotation – balanced!} \]
Single rotation example: \texttt{insert(1)}

\begin{itemize}
  \item Mirror image to left-left case, so you rotate the other way
  \begin{itemize}
    \item Single rotation to the left
  \end{itemize}
  \end{itemize}

\begin{itemize}
  \item Exact same concept, but slightly different code
\end{itemize}

Another example: \texttt{insert(16)}

The general right-right case

- Mirror image to left-left case, so you rotate the other way
  - Single rotation to the left
  - Exact same concept, but slightly different code

\begin{itemize}
  \item After insertion – unbalanced!
  \item After single rotation – balanced!
\end{itemize}
Bad Case #3

Insert(1)
Insert(6)
Insert(3)

Bad Case #3: Wrong Solution #1

Unfortunately, single rotations are not enough for insertions in the left-right subtree or the right-left subtree

Simple example: \text{insert}(1), \text{insert}(6), \text{insert}(3)

- First wrong idea: single rotation like we did for left-left

\[ \text{Doesn’t work!!!} \]

Bad Case #3: Correct Solution: Double Rotation

AVL Property violated at this node (“x”)

1. Rotate between x’s child and grandchild
2. Rotate between x and x’s new child

Double Rotation
General right-left case: Double Rotation

The general right-left case (cont.)

- Like in the left-left and right-right cases, the height of the subtree after rebalancing is the same as before the insert
  - So no ancestor in the tree will need rebalancing
- Does not have to be implemented as two rotations; can just do: After insertion – unbalanced! After entire double rotation – balanced!

The last case: left-right

- Mirror image of right-left – double rotation
  - Again, no new concepts, only new code to write

Double rotation: **insert (5)**, step 1
Double rotation: \textbf{insert (5), step 2}

AVL Insert - Summary

- Insert as in a BST
- Check back up path for imbalance, which will be 1 of 4 cases:
  - node’s left-left grandchild is too tall
  - node’s left-right grandchild is too tall
  - node’s right-left grandchild is too tall
  - node’s right-right grandchild is too tall
- Only one case occurs because tree was balanced before insert
- After the appropriate single or double rotation, the smallest-unbalanced subtree has the same height as before the insertion
  - So all ancestors are now balanced

Imbalance at node X

Single Rotation
1. Rotate between x and child

Double Rotation
1. Rotate between x’s child and grandchild
2. Rotate between x and x’s new child

Insert into an AVL tree: \text{a b e c d}
Single and Double Rotations:

Inserting what integer values would cause the tree to need a:

1. single rotation?

2. double rotation?

3. no rotation?

Student Activity

Insert 3

Insert(3)

Unbalanced?

Insert 33

Insert(33)

Unbalanced?

How to fix?

Insert 33: Single Rotation

Insert(33): Single Rotation
How to fix?

Unbalanced?

How to fix?

Insert 18

Insert 18: Single Rotation (oops!)

Insert 18: Double Rotation (Step #1)

Insert 18: Double Rotation (Step #2)
AVL Trees Revisited

- **Balance condition:**
  - For every node \( x \), \(-1 \leq \text{balance}(x) \leq 1\)
  - Strong enough: Worst case depth is \( O(\log n) \)
  - Easy to maintain: one single or double rotation

- **Guaranteed \( O(\log n) \) running time for**
  - Find ?
  - Insert ?
  - Delete ?
  - buildTree ?

AVL Trees Revisited

- **What extra info** did we maintain in each node?

- **Where** were rotations performed?

- **How did we locate** this node?