Minimum Spanning Tree

CSE 373
Data Structures
Unit 15

Reading: Chapter 9.5

Example of a Spanning Tree

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>21</td>
<td>75</td>
<td>64</td>
</tr>
<tr>
<td>25</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>18</td>
<td>80</td>
<td>4</td>
</tr>
</tbody>
</table>

Price of this tree = 18+19+4+10+17+64

Minimum Spanning Tree

- Each edge has a cost.
- Find a minimal-cost subset of edges that will keep the graph connected. (must be a ST).

![Graph](image)

Minimum Spanning Tree Problem

- Input: Undirected connected graph $G = (V,E)$ and a cost function $C$ from $E$ to the reals. $C(e)$ is the cost of edge $e$.
- Output: A spanning tree $T$ with minimum total cost. That is: $T$ that minimizes

$$C(T) = \sum_{e \in T} C(e)$$

- Another formulation: Remove from $G$ edges with maximal total cost, but keep $G$ connected.
Minimum Spanning Tree

- Boruvka 1926
- Kruskal 1956
- Prim 1957 also by Jarnik 1930
- Karger, Klein, Tarjan 1995
  - Randomized linear time algorithm
  - Probably not practical, but very interesting

An Algorithm for MST

- The algorithm colors the edges of the graph. Initially, all edges are black.
- A blue edge - belongs to T.
- A red edge - does not belong to T.
- We continue to color edges until we have n-1 blue edges.
- How do we select which edge to color next? How do we select its color?

Minimum Spanning Tree Problem

- Definition: For a given partition of V into U and V-U, the cut defined by U is the set of edges with one end in U and one end in V-U.

The cut defined by U

The Blue/Red Edge-coloring Rules

- The blue rule: Find a cut with no blue edge. Color blue the cheapest black edge in the cut.
- The red rule: Find a cycle with no red edge. Color red the most expensive black edge in the cycle.

These rules can be applied in any order. We will see two specific algorithms.
Example of Blue/Red rules (1)

Consider the cut defined by \{2,3\}
- color (1,2) blue

Example of Blue/Red rules (2)

Consider the cycle (7-5-4)
- color (4,5) red

Example of Blue/Red rules (3)

Consider the cut defined by \{3,5,6\}
- color (5,7) blue

Example of Blue/Red rules (4)

Consider the cycle (1-2-7-5)
- color (1,5) red
Example of Blue/Red rules (5)

Consider the cycle (1-2-7-5-6)
- color (2,7) red.

Example of Blue/Red rules (6)

Consider the cut defined by {4}
- color (4,7) blue

Example of Blue/Red rules (7)

Consider the cut defined by {6}
- color (5,6) blue

Example of Blue/Red rules (8)

Consider the cut defined by {3}
- color (2,3) blue
Example of Blue/Red rules (9)

Consider the cut defined by \{1,2,3\}
- color \{(1,6)\} blue

Example of Blue/Red rules (10)

Final MST

Proof of Blue/Red Rules

• Claim: for any \(k \geq 0\), after we color \(k\) edges there exists an MST that includes all the blue edges and none of the red edges.
• Proof: By induction on \(k\).
• Base: \(k=0\) trivially holds.
• Step: Assume this is true after we color \(k-1\) edges \(e_1, e_2, ..., e_{k-1}\). Consider the coloring of \(e_k\).

Case 1: Applying the Blue Rule

\(C(u,v)\) is minimal
Case 1: Applying the Blue Rule

If \((u,v) \in T\), then \(T\) must includes some other edge \((x,y)\) in the cut defined by \(U\) (\(T\) is connected, so there is a path \(u-v\)).

Case 2: Applying the Red Rule

Assume \((u,v) \in T\).
By removing \((u-v)\) from \(T\) we get two components.

Case 1: Applying the Blue Rule

Consider \(T' = T \cup (u,v) - (x,y)\)

\[ C(T') = C(T) + C(u,v) - C(x,y) \]
\[ C(T') \leq C(T) \]
\(T'\) is also a minimum spanning tree, and it includes \(e_k\)

Case 2: Applying the Red Rule

The cycle that causes us to color \((u-v)\) red includes an edge connecting the two component (whose cost is at most \(c(u,v)\)).

There is an alternative MST, that does not include \(e_k\)
One more point: We can always proceed

Select an edge $e$.

- If $e$ connects two blue sub-trees, then there is a cut without any blue edge and we can run the blue rule on this cut.
- Otherwise, $e$ closes a cycle in which $e$ is the most expensive edge (why?) so we can color $e$ red.

Kruskal’s Greedy Algorithm

Sort the edges by increasing cost;
Initialize $T$ to be empty;
For each edge $e$ chosen in increasing order do
  if adding $e$ does not form a cycle then
    add $e$ to $T$

Proof: The algorithm follows the blue/red rules:
- If $e$ closes a cycle - apply the red rule (by the sorting, $e$ is the most expensive in this cycle).
- Otherwise - apply the blue rule ($e$ connects two components, consider the cut defined by any of them. $e$ is the cheapest edge in this cut)

Example of Kruskal 1

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
\hline
2 & 3 & 0 & 4 & 1 & 2 & 2 & 2 \\
3 & 4 & 1 & 5 & 2 & 6 & 7 & 7 \\
0 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\
\end{array}
\] 

\{7,4\} \{2,1\} \{7,5\} \{5,6\} \{5,4\} \{1,6\} \{2,7\} \{2,3\} \{3,4\} \{1,5\}

Example of Kruskal 2

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\
\hline
2 & 3 & 0 & 4 & 1 & 2 & 2 & 2 \\
3 & 4 & 1 & 5 & 2 & 6 & 7 & 7 \\
0 & 1 & 1 & 2 & 2 & 3 & 3 & 3 \\
\end{array}
\] 

\{7,4\} \{2,1\} \{7,5\} \{5,6\} \{5,4\} \{1,6\} \{2,7\} \{2,3\} \{3,4\} \{1,5\}
Example of Kruskal 2

Example of Kruskal 3

Example of Kruskal 4

Example of Kruskal 5
Data Structures for Kruskal

- Sorted edge list
  \[\{7,4\} \{2,1\} \{7,5\} \{5,6\} \{5,4\} \{1,6\} \{2,7\} \{2,3\} \{3,4\} \{1,5\}\]
  0 1 1 2 2 3 3 3 3 4

- Disjoint Union / Find
  - Union(a, b) - union the disjoint sets named by a and b
  - Find(a) returns the name of the set containing a

Remark: The set name is one of its members

Example of DU/F (1)

\[\{7,4\} \{2,1\} \{7,5\} \{5,6\} \{5,4\} \{1,6\} \{2,7\} \{2,3\} \{3,4\} \{1,5\}\]

Find(5) = 7
Find(4) = 7

Example of DU/F (2)

\[\{7,4\} \{2,1\} \{7,5\} \{5,6\} \{5,4\} \{1,6\} \{2,7\} \{2,3\} \{3,4\} \{1,5\}\]

Find(1) = 1
Find(6) = 7

Example of DU/F (3)

\[\{7,4\} \{2,1\} \{7,5\} \{5,6\} \{5,4\} \{1,6\} \{2,7\} \{2,3\} \{3,4\} \{1,5\}\]

Union(1,7)

u, v in the same set (u, v) is not added to T

u, v in different sets add (u, v) to T, union the sets.
Kruskal’s Algorithm with DU / F

Sort the edges by increasing cost;
Initialize T to be empty;
for each edge \{(i, j)\} chosen in increasing order do
  u := Find(i);
  v := Find(j);
  if \(u \neq v\) then
    add \{(i, j)\} to T;
    Union(u, v);

Amortized Complexity

- Disjoint union/find can be implemented such that the average time per operation is essentially a constant.
- An individual operation can be costly, but over time the average cost per operation is not.
- On average, each U/F operation takes \(O(m \alpha(m, n))\) time.

Ekerman function. Practically, this is a constant.

Evaluation of Kruskal

- \(G\) has \(n\) vertices and \(m\) edges.
- Sort the edges - \(O(m \log m)\).
- Traverse the sorted edge list using efficient UF - \(O(m \alpha(m, n))\).
- Total time is \(O(m \log m)\).

Prim’s Algorithm

- We maintain a single tree.
- Initially, the tree consists of one vertex.
- For each vertex not in the tree maintain the cheapest edge to a vertex in the tree (if exists).
**Correctness Proof for Prim**

- Repeatedly executes the blue rule \((n-1)\) times.
- In each step we consider the cut defined by the vertices that are already in \(T\).

**Data Structures for Prim**

- Adjacency Lists - we need to look at all the edges from a newly added vertex.
- Array for the best edges to the tree.
Data Structures for Prim

- Priority queue for all edges to the tree (orange edges).
  - Insert, delete-min, delete (e.g. binary heap).

Evaluation of Prim

- \( n \) vertices and \( m \) edges.
- Priority queue \( O(\log n) \) per operation.
- \( O(m) \) priority queue operations.
  - An edge is visited when a vertex incident to it joins the tree.
- Time complexity is \( O(m \log n) \).
- Storage complexity is \( O(m) \).

Kruskal vs Prim

- Kruskal
  - Simple
  - Good with sparse graphs - \( O(m \log m) \)

- Prim
  - More complicated
  - Perhaps better with dense graphs - \( O(m \log n) \)

Note: \( O(\log n) = O(\log m) \) (since \( m < n^2 \))