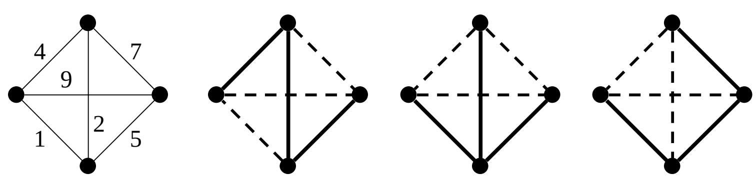
Lecture 25: Kruskal and beyond…

- <u>What we will munch on today:</u>
  - Minimum Spanning Trees
    - Prim's Algorithm
    - Kruskal's Algorithm
  - Those Puzzles from 4th grade!
  - Euler Circuits and Tours
- Covered in Chapter 9 in the textbook

Recall from Last Time: Spanning Trees

- *Spanning tree*: subset of edges from a connected graph G = (*V*,*E*) that:
  1. touches all vertices in the graph (*spans* the graph), and
  2. forms a tree (is connected, with no cycles → |V|-1 edges)



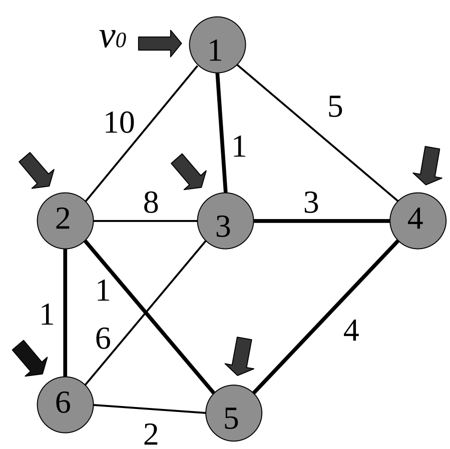
- *Minimum spanning tree (MST)*: spanning tree with the least total edge cost

Why greed works for finding MSTs…

- For any spanning tree T, inserting an edge e not in T creates a cycle → Removing any edge gives back a spanning tree
  - If e had a lower cost than removed edge, we get a lower cost spanning tree
- <u>Idea</u>: Create a spanning tree as follows:
  1. Add an edge of minimum cost that doesn't create a cycle
  2. Repeat Step 1 for $|V|$-1 edges
- This spanning tree has minimum cost because:
  - if you can replace an edge with another edge of lower cost without creating a cycle, our algorithm would have picked it
- <u>Two MST algorithms</u>: Prim (1957) and Kruskal, Jr. (1956)
  - Differ in how an edge of minimum cost is picked

Prim's Algorithm for Finding the MST

1. Starting from an empty tree, *T*, pick a vertex, *v0*, at random and initialize: *V'* = {*v0*} and *E'* = {}
2. Choose a vertex *v* not in *V'* such that *edge weight from v to a vertex in V' is minimal* (get greedy!)
3. Add *v* to *V'* and the edge to *E'* if no cycle is created
4. Repeat until all vertices have been added



Prim's Algorithm: Implemented and Analyzed

- Implementation details:
  1. Initialize cost of each node to ∞ and mark it unknown
  2. Initialize cost of one selected node S to 0, with Prev[S] = 0
  3. While there are unknown nodes left in the graph
     1. Select the unknown node *N* with the *lowest cost*
     2. Mark *N* as known
     3. For each unknown node *A* adjacent to *N*
        If cost of (*N*, *A*) < *A*'s cost
          A's cost = cost of (*N*, *A*)
          Prev[*A*] = *N* *//store preceding node*
- This is almost identical to Dijkstra's algorithm!
- Run time is $O(|V|^2)$ without heaps and $O(|V| \log |V| + |E| \log |V|)$ using binary heaps

Kruskal finds another greedy way to MST

- In 1956, J. B. Kruskal, Jr. found another way to find MSTs
- Main Idea: Select edges in order of increasing cost and accept an edge only if it does not cause a cycle
- Pseudocode for Kruskal's MST algorithm:
  - Put all the vertices into single node trees by themselves
  - Put all the edges in a priority queue with key = edge cost
  - Repeat until |V|-1 edges have been accepted
    - *Extract cheapest edge*
    - If it forms a cycle, ignore it
      else accept the edge – it will join two existing trees and yield a larger tree
  - Return the accepted edges (they form the spanning tree)

Kruskal’s Algorithm in C

```
Forest Kruskal_MST( Graph g, int n, double **costs )
{
   Forest T;
   Queue q;
   Edge e;
   T = ConsForest( g );
   q = BuildHeap( g, costs );
   for(i=0;i<(n-1);i++) {
       do {
          e = DeleteMin( q );
       } while ( Cycle( e, T ) );
       AddEdge( T, e );
   }
   return T;
}
```

n = |*V*|

Initial Forest: single vertex trees

Priority Q of edges

Kruskal’s Algorithm in C

```
Forest Kruskal_MST( Graph g, int n, double **costs )
{
   Forest T;
   Queue q;
   Edge e;
   T = ConsForest( g );
   q = BuildHeap( g, costs );
   for(i=0;i<(n-1);i++) {
       do {
          e = DeleteMin( q );
       } while ( Cycle( e, T ) );
       AddEdge( T, e );
   }
   return T;
}
```

Need n-1 edges to fully connect (span) n vertices

Kruskal's Algorithm in C

```
Forest Kruskal_MST( Graph g, int n, double **costs )
{
    Forest T;
    Queue q;
    Edge e;
    T = ConsForest( g );
    q = BuildHeap( g, costs );
    for(i=0;i<(n-1);i++) {
        do {
            e = DeleteMin( q );
        } while ( Cycle( e, T ) );
        AddEdge( T, e );
    }
    return T;
}
```

Try the cheapest edge

Until we find one that doesn't form a cycle

... and add it to the forest!

Kruskal's Algorithm in C

```
Forest Kruskal_MST( Graph g, int n, double **costs )
{
    Forest T;
    Queue q;
    Edge e;
    T = ConsForest( g );
    q = BuildHeap( g, costs );
    for(i=0;i<(n-1);i++) {
        do {
            e = DeleteMin( q );
        } while ( Cycle( e, T ) );
        AddEdge( T, e );
    }
    return T;
}
```

But how do we detect a cycle?

Hints for Detecting Cycles in Kruskal's Method

- Initially, you have n different elements (single vertex trees)
- After you have added some edges, you have fewer elements – several disconnected trees, each with a subset of vertices
- When do you get a cycle? If you add an edge (u,v) where both u and v are already in the same tree T_i, you get a cycle
  - Therefore, to check for cycles, you only need to find out if u and v are in the same tree
  - If not, then the edge can be added and we union vertices in u's tree with vertices in v's tree
- What is your favorite data structure for such operations?

Disjoint Set ADT in Kruskal's Algorithm

- Here's how the disjoint set ADT makes an appearance:
  - In Kruskal's algorithm, connected vertices form equivalence classes (they are in the same tree)
    - "Being connected" is the equivalence relation
- Initially, each vertex is in a class by itself
- As edges are added, more vertices become related and the equivalence classes grow
- Until finally all the vertices are in a single equivalence class

Union/Find in Kruskal's Algorithm

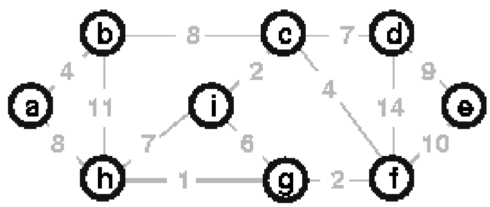
- Representatives
  - One vertex in each class can be the representative of that class
  - Vertices can be stored in up-tree data structures with roots = class representatives
    - This is what we used for Union-Find
- Detecting cycles is easy!
  - For each edge (u,v) that you're going to add
    - If Find(u) == Find(v), then u and v are in the same class (same tree) and therefore the edge will form a cycle
    - Otherwise, we accept the edge and do Union(u,v)

Kruskal's Algorithm in C

```
Forest Kruskal_MST( Graph g, int n, double **costs ) {
   Forest T;
   Queue q;
   Edge e;
   DisjSet S = InitializeSet( g );
   T = ConsForest( g );
   q = BuildHeap( g, costs );
   for(i=0;i<(n-1);i++) {
       do {
          e = DeleteMin( q ); // e = (u,v)
       } while ( (Find(u,S) == Find(v,S)) );
       AddEdge( T, e );
       Union(S, u, v);
   }
   return T;  }
```

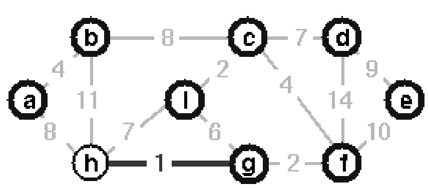
Kruskal in action

All the vertices are in single element trees



Each vertex is its own representative

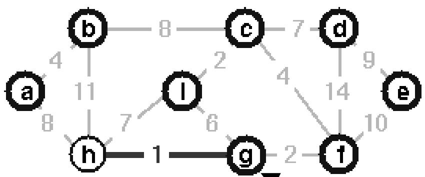
Kruskal in action



The cheapest edge is h-g

Add it to the forest, joining h and g into a 2-element tree

Kruskal in action



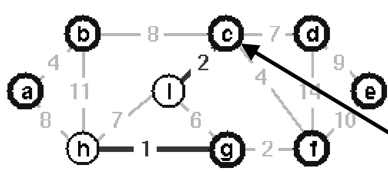
Add it to the forest, joining **h** and **g** into a 2-element tree

Choose **g** as its representative

Kruskal in action

The next cheapest edge is **c-i**

Add it to the forest, joining **c** and **i** into a 2-element tree



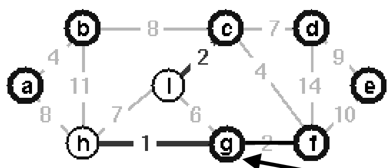
Choose **c** as its representative

Our forest now has 2 two-element trees and 5 single vertex ones

Kruskal in action

The next cheapest edge is g-f

Add it to the forest, joining g and f into a 3-element tree



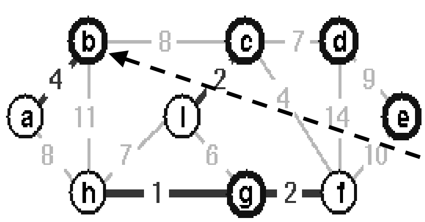
Choose g as its representative

Our forest now has 1 three-element tree, 1 two-element tree, and 4 single vertex ones

Kruskal in action

The next cheapest edge is a-b

Add it to the forest, joining a and b into a 2-element tree



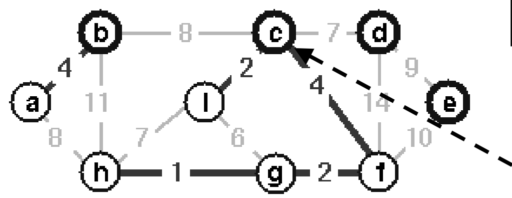
Choose b as its representative

Our forest now has only 2 single vertex trees

Kruskal in action

The next cheapest edge is c-f

Add it to the forest: *Merge* two 2-element trees (Union the 2 sets)



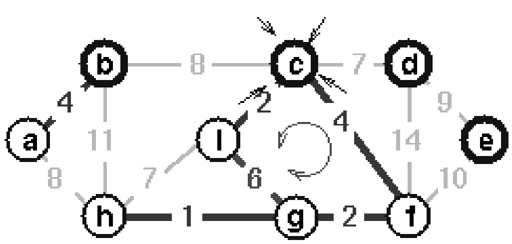
Choose the rep of one as its representative

Kruskal in action

The next cheapest edge is g-i

Find(g) is c

Find(i) is also c

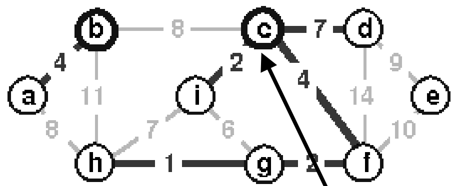


g-i forms a cycle!

Ignore this edge

Kruskal in action

The next cheapest edge is **c-d**



Find(**c**) is **c**

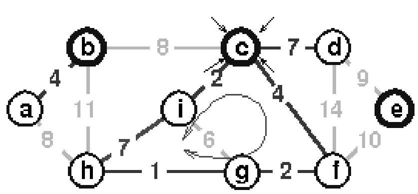
Find(**d**) is **d**

c-d joins two trees, so we add **d**

.. and keep **c** as the representative

Kruskal in action

The next cheapest edge is **h-i**



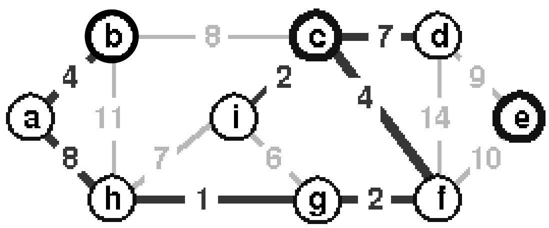
Find(**h**) is **c**

Find(**i**) is **c**

h-i forms a cycle, so ignore it!

Kruskal in action

The next cheapest edge is a-h



Find(a) is b

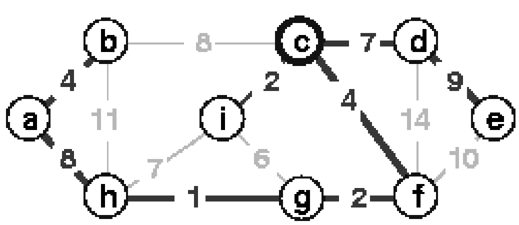
Find(h) is c

a-h joins two trees, and we add it

Kruskal done!

The next cheapest edge is b-c

But b-c forms a cycle



So add d-e instead

That's n-1 edges added – we now have a spanning tree!

Kruskal's Algorithm: Analysis

```
Forest Kruskal_MST( Graph g, int n, double **costs ) {
   Forest T;
   Queue q;
   Edge e;
   DisjSet S = InitializeSet( g );
   T = ConsForest( g );
   q = BuildHeap( g, costs );
   for(i=0;i<(n-1);i++) {
       do {
          e = DeleteMin( q ); // e = (u,v)
       } while ( (Find(u,S) == Find(v,S)) );
       AddEdge( T, e );
       Union(S, u, v);
   }
   return T;  }
```

InitializeSet, ConsForest: O(|V|)

BuildHeap: O(|E|)

Worst case: DeleteMin |E| edges, each O(log |E|)

AddEdge: O(1)

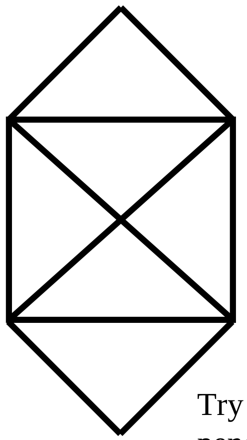
Find, Union: ≈ O(1) amortized

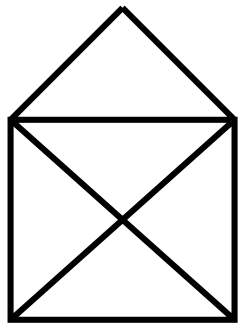
Total time = O(|E| log |E|)

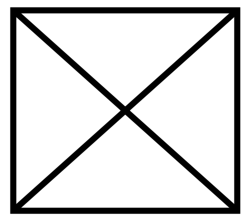
Kruskal versus Prim

- Worst case running time
  - Prim: O(|V| log |V| + |E| log |V|)
  - Kruskal: O(|E| log |E|) = O(|E| log |V|) since |E| = O(|V|^2)
- Kruskal usually runs much faster than O(|E| log |V|) in practice
  - Not all edges need to be DeleteMin-ed typically
  - The required |V|-1 edges are usually found quickly
  - So, Kruskal tends to be faster than Prim

It’s Puzzle Time!

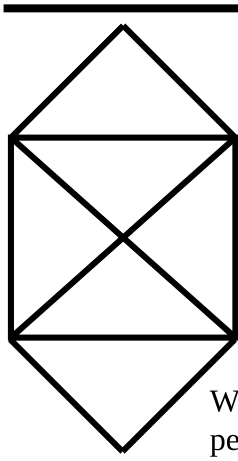


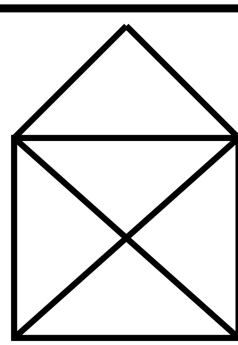


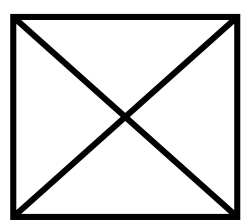


Try drawing these figures without lifting your pencil, drawing each line only once…
(begin: memories of 4th grade days…)

It’s Puzzle Time!

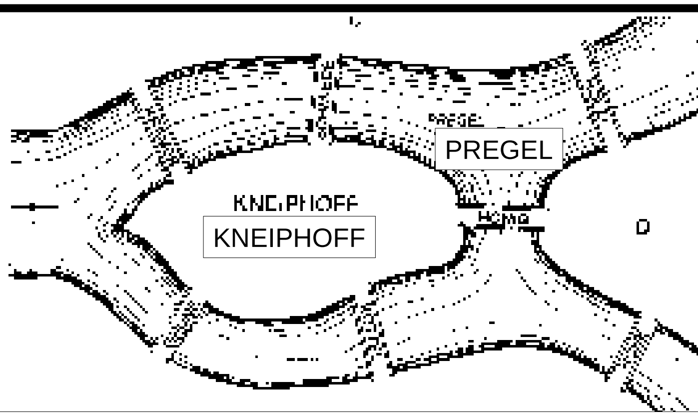






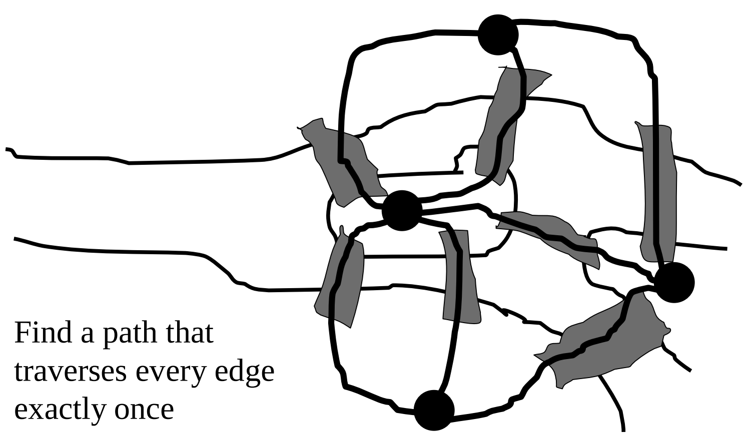
Which of these can you draw without lifting your pencil, drawing each line only once?
Can you start and end at the same point?
(end: memories of 4th grade days…)

Historical Puzzle: Seven Bridges of Königsberg



Want to cross all bridges but…
Can cross each bridge only once (High toll to cross twice?!)

A “Multigraph” for the Bridges of Königsberg



Find a path that traverses every edge exactly once

Euler Circuits and Tours

- Euler tour: a path through a graph that visits each edge exactly once
- Euler circuit: an Euler tour that starts and ends at the same vertex
- Named after Leonhard Euler (1707-1783), who cracked this problem and founded graph theory in 1736
- Some observations for undirected graphs:
  - An Euler circuit is only possible if the graph is connected and each vertex has even degree (= # of edges on the vertex) [Why?]
  - An Euler tour is only possible if the graph is connected and either all vertices have even degree or exactly two have odd degree [Why?]

Euler Circuit Problem

- Problem: Given an undirected graph G = (*V*,*E*), find an Euler circuit in G
- Note: Can check if one exists in linear time (how?)
- Given that an Euler circuit exists, how do we *construct* an Euler circuit for G?
- Hint: Think deep! We've discussed the answer in depth before…

Next Class:

Constructing Euler circuits

The vast gulf between Euler and Hamiltonian circuits

The dreaded world of NP hardness

To Do:

Programming Assignment #2 (Due in 6 days!!)

Finish reading chapter 9 (and have a great weekend!)