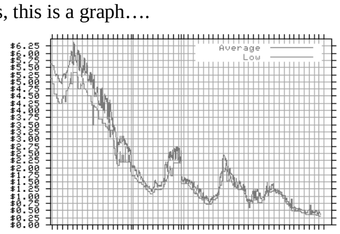
Lecture 22: Let’s Get Graphic – Graph Algorithms

- Today’s Agenda:
  - What is a graph?
  - Some graphs that you already know
  - Definitions and Properties
  - Implementing Graphs
  - Topological Sort

- Covered in Chapter 9 of the textbook

What are graphs? (Take 1)

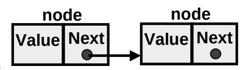
- Yes, this is a graph….

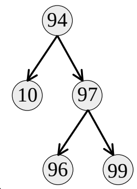


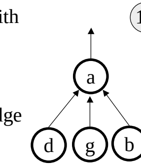
- But we are interested in a different kind of “graph”

Motivation for Graphs

- Consider the data structures we have looked at so far…
- Linked list: nodes with 1 incoming edge + 1 outgoing edge
- Binary trees/heaps: nodes with 1 incoming edge + 2 outgoing edges
- Binomial trees/B-trees: nodes with 1 incoming edge + multiple outgoing edges
- Up-trees: nodes with multiple incoming edges + 1 outgoing edge



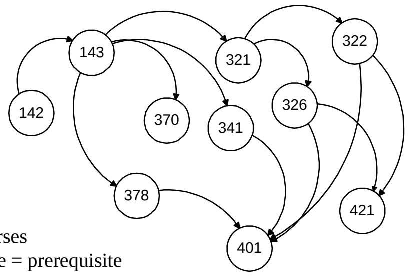




Motivation for Graphs

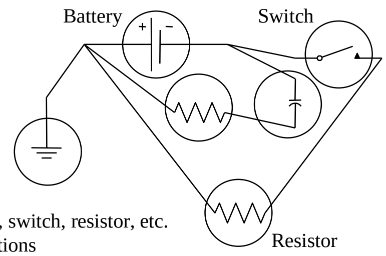
- What is common among these data structures?
- How can you generalize them?
- Consider data structures for representing the following problems…

Course Prerequisites for CSE at UW



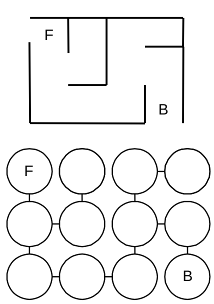
Nodes = courses
Directed edge = prerequisite

Representing Electrical Circuits



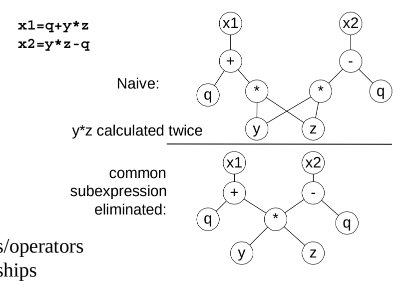
Nodes = battery, switch, resistor, etc.
Edges = connections

Representing the Floor Plan of a House



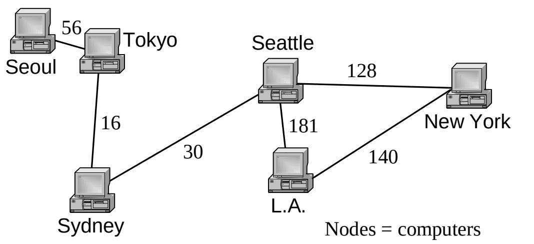
Nodes = rooms
Edge = door or passage

Representing Expressions in Compilers



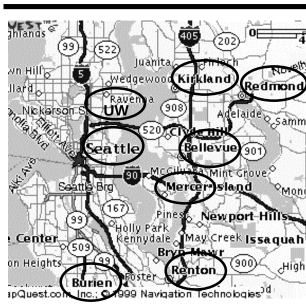
Nodes = symbols/operators
Edges = relationships

Information Transmission in a Computer Network



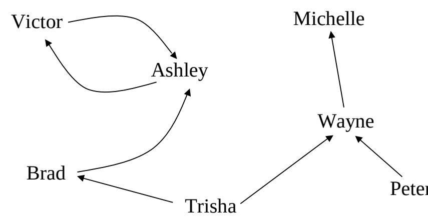
Nodes = computers
Edges = transmission rates

Traffic Flow on Highways

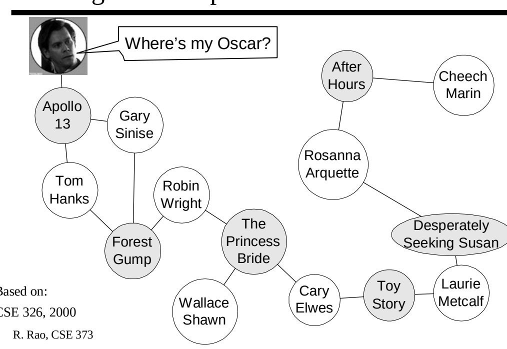


Nodes = cities
Edges = # vehicles on connecting highway

Soap Opera Relationships

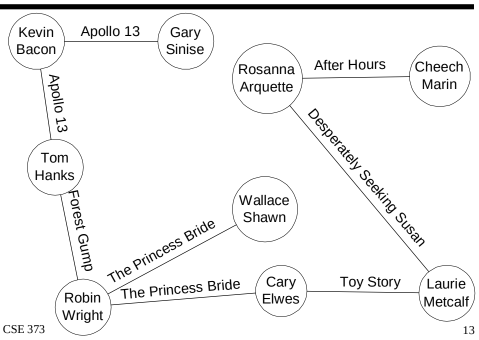


Six Degrees of Separation from Kevin Bacon



Based on:
CSE 326, 2000

Six Degrees of Separation from Kevin Bacon



Graphs: Definition

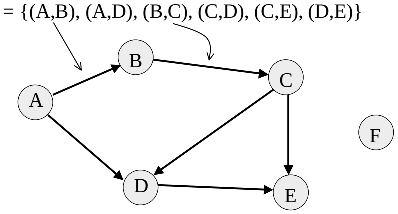
- A graph is simply a collection of nodes plus edges
  - Linked lists, trees, and heaps are all special cases of graphs
- The nodes are known as vertices (node = "vertex")
- Formal Definition: A graph G is a pair (V, E) where
  - V is a set of vertices or nodes
  - E is a set of edges that connect vertices

Graph Example

- Here is a graph $G = (V, E)$
  - Each <u>edge</u> is a pair (v_1, v_2), where v_1, v_2 are vertices in V

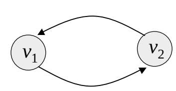
  V = {A, B, C, D, E, F}

  E = {(A,B), (A,D), (B,C), (C,D), (C,E), (D,E)}

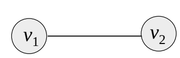


Directed versus Undirected Graphs

- If the order of edge pairs (v_1, v_2) matters, the graph is directed (also called a digraph): $(v_1, v_2) \neq (v_2, v_1)$



- If the order of edge pairs (v_1, v_2) does not matter, the graph is called an undirected graph: in this case, $(v_1, v_2) = (v_2, v_1)$



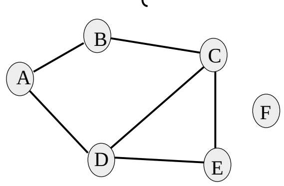
Graph Representations

- Space and time are measured in terms of:
  - Number of vertices = $|V|$ and
  - Number of edges = $|E|$
- There are two ways of representing graphs:
  - The *adjacency matrix* representation
  - The *adjacency list* representation

Graph Representation: Adjacency Matrix

The *adjacency matrix* representation: Space = ?

$$M(v, w) = \begin{cases} 1 & \text{if } (v, w) \text{ is in E} \\ 0 & \text{otherwise} \end{cases}$$

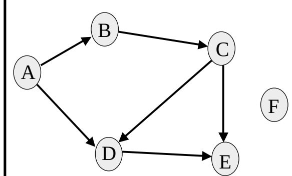


$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Adjacency Matrix for a Digraph

Space = $|V|^2$

$$M(v, w) = \begin{cases} 1 & \text{if } (v, w) \text{ is in E} \\ 0 & \text{otherwise} \end{cases}$$

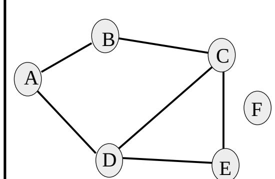


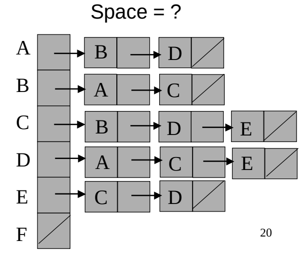
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Graph Representation: Adjacency List

The *adjacency list* representation: For each v in V, $L(v)$ = list of w such that (v, w) is in E

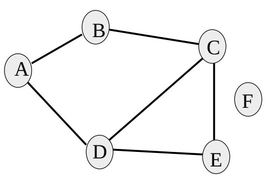
Space = ?

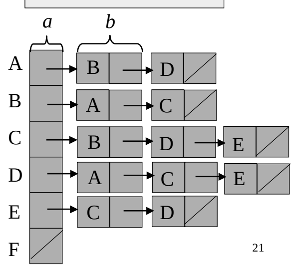




Graph Representation: Adjacency List

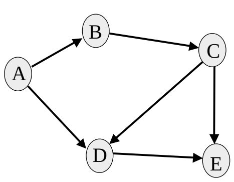
Space = a |V| + 2 b |E|



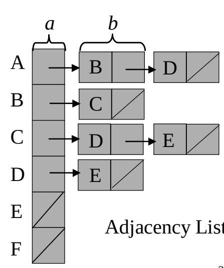


Adjacency List for a Digraph

Space = ?



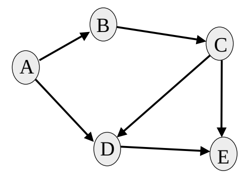
Digraph



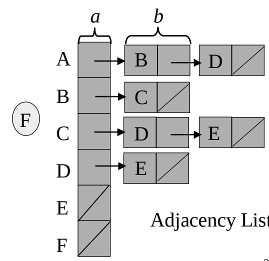
Adjacency List

Adjacency List for a Digraph

Space = a |V| + b |E|



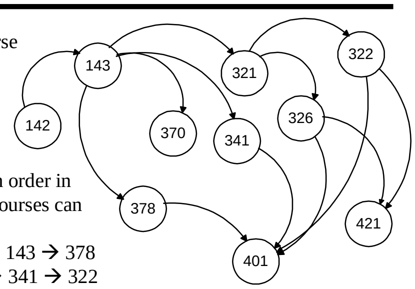
Digraph



Adjacency List

Graph Algorithm #1: Topological Sort

Graph of course prerequisites

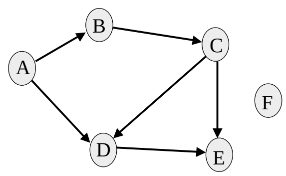


Problem: Find an order in which all these courses can be taken.
Example: 142 → 143 → 378 → 370 → 321 → 341 → 322 → 326 → 421 → 401

To take a course, all its prerequisites must be taken first

Topological Sort

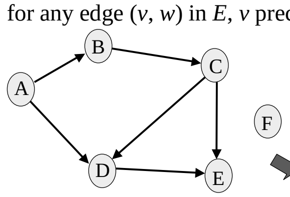
Topological sorting problem: given digraph $G = (V, E)$, find a linear ordering of its vertices such that:
<u>for any edge (v, w) in E, v precedes w in the ordering</u>



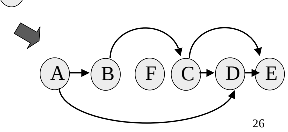
On-board example:
Topo-Sort this digraph

Topological Sort

Topological sorting problem: given digraph $G = (V, E)$, find a linear ordering of its vertices such that:
for any edge (v, w) in E, v precedes w in the ordering

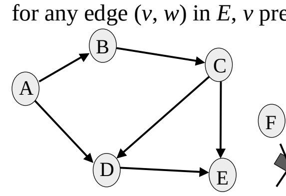


Any linear ordering in which all the arrows go to the right is a valid solution

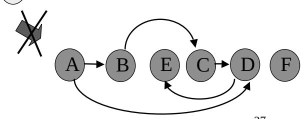


Topological Sort

Topological sorting problem: given digraph $G = (V, E)$, find a linear ordering of its vertices such that:
for any edge (v, w) in E, v precedes w in the ordering

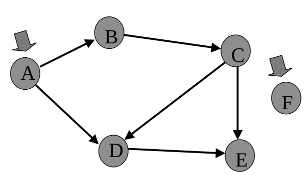


Not a valid topological sort!



Topological Sort Algorithm #1

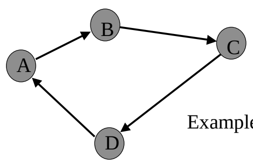
<u>Step 1</u>: Identify vertices that have no incoming edges
- The "in-degree" of these vertices is zero



Topological Sort Algorithm #1

<u>Step 1</u>: Identify vertices that have no incoming edges

- If *no such vertices*, graph has <u>cycle(s)</u> (cyclic graph)
- Topological sort not possible – Halt.

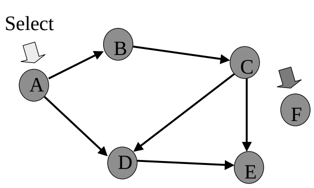


Example of a cyclic graph

Topological Sort Algorithm #1

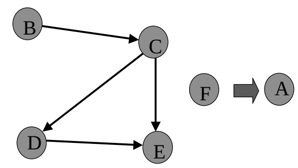
<u>Step 1</u>: Identify vertices that have no incoming edges

- Select one such vertex



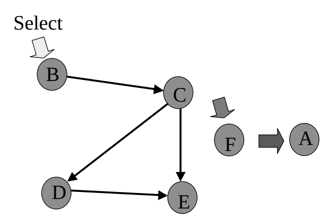
Topological Sort Algorithm #1

<u>Step 2</u>: Delete this vertex of in-degree 0 and all its outgoing edges from the graph. Place it in the output.



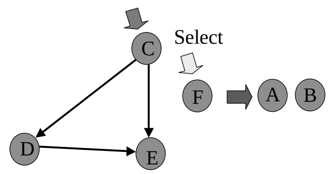
Topological Sort Algorithm #1

Repeat <u>Step 1</u> and <u>Step 2</u> until graph is empty



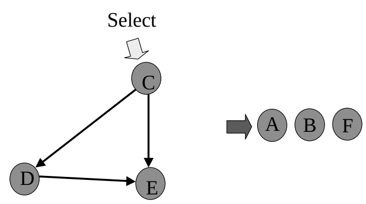
Topological Sort Algorithm #1

Repeat Step 1 and Step 2 until graph is empty



Topological Sort Algorithm #1

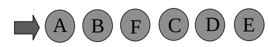
Repeat Step 1 and Step 2 until graph is empty



Topological Sort Algorithm #1

Repeat Step 1 and Step 2 until graph is empty

Final Result:



Topological Sort Algorithm #1: Analysis

For input graph G = (V,E), Run Time = ?

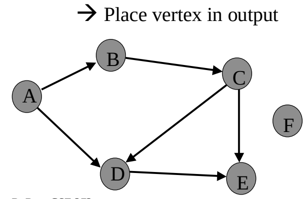
Break down into total time to:

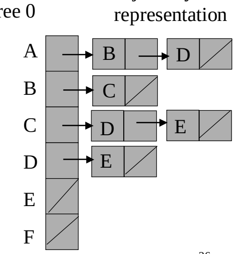
→ Find a vertex with in-degree 0

→ Remove its edges

→ Place vertex in output

Assume adjacency list representation



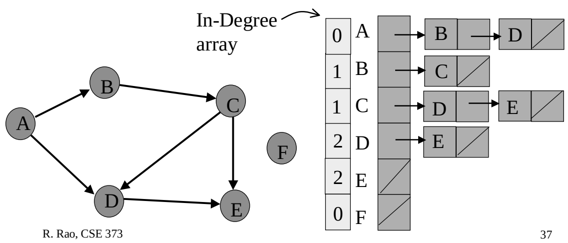


Topological Sort Algorithm #1: Analysis

Calculate and store In-Degree of all vertices in an array

→ Find vertex with in-degree 0: Search this array

→ Remove its edges: Update this array



Topological Sort Algorithm #1: Analysis

For input graph G = (V,E), Run Time = ?

Break down into total time to:

→ Find vertices with in-degree 0: $|V|$ vertices, each takes $O(|V|)$ to search In-Degree array = $O(|V|^2)$

→ Remove edges: $|E|$ edges

→ Place vertices in output: $|V|$ vertices

Total time = $\mathbf{O(|V|^2 + |E|)}$

Can we do better than quadratic time?

Can you think of a faster way to find vertices with in-degree 0?

Next Class: Faster Topological Sort and

Finding shortest ways to get to your classrooms

To Do:

Read and enjoy chapter 9

Have a great weekend!