CSE 373 Lecture 11: Binomial Queues

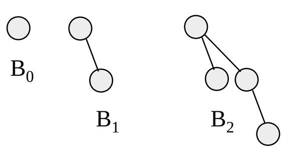
- Today's Topics:
  - Binomial Queues
    - Merge
    - Insert
    - DeleteMin
    - Implementation
  - Other Priority Queues: d-heaps, leftist, and skew heaps
- Covered in Chapter 6 in the text

Binomial Queues

- Binomial queues support all three priority queue operations Merge, Insert and DeleteMin in O(log N) time
- Idea: Maintain a collection of heap-ordered trees
  - *Forest of binomial trees*
- Recursive Definition of Binomial Tree (based on height k):
  - Only one binomial tree for a given height
  - Binomial tree of height 0 = single root node
  - Binomial tree of height k = B_k = Attach B_{k-1} to root of another B_{k-1}

3 Steps to Building a Binomial Tree

- To construct a binomial tree B_k of height k:
  1. Take the binomial tree B_{k-1} of height k-1
  2. Place another copy of B_{k-1} one level below the first
  3. Attach the root nodes

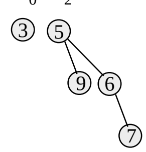


- Binomial tree of height k has exactly 2^k nodes (by induction)

Definition of Binomial Queues

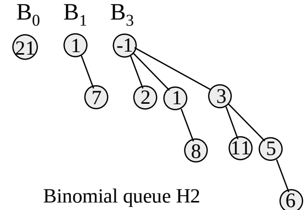
Binomial Queue = "forest" of heap-ordered binomial trees

B_0 B_2



Binomial queue H1
5 elements = 101 base 2
→ B_2 B_0

B_0 B_1 B_3



Binomial queue H2
11 elements = 1011 base 2
→ B_3 B_1 B_0

Binomial Queue Properties

- Suppose you are given a binomial queue of N nodes

1. There is a unique set of binomial trees for N nodes
2. What is the maximum number of trees that can be in an N-node queue?
   - 1 node → 1 tree B_0; 2 nodes → 1 tree B_1; 3 nodes → 2 trees B_0 and B_1; 7 nodes → 3 trees B_0, B_1 and B_2 …

Number of Trees in a Binomial Queue

- What is the maximum number of trees that can be in an N-node binomial queue?
  - 1 node → 1 tree B_0; 2 nodes → 1 tree B_1; 3 nodes → 2 trees B_0 and B_1; 7 nodes → 3 trees B_0, B_1 and B_2 …
- Trees B_0, B_1, …, B_k can store up to $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$ nodes = N.
- Maximum is when all trees are used.
- So, number of trees in an N-node binomial queue is ≤ (log(N+1)-1)+1= O(log N)

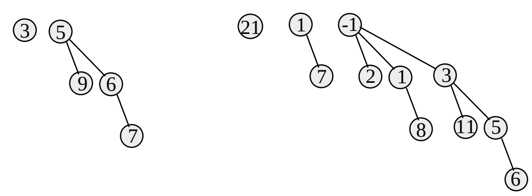
Binomial Queues: Merge

- Main Idea: Merge two binomial queues by merging individual binomial trees
  - Since B_{k+1} is just two B_k's attached together, merging trees is easy
- Steps for creating new queue by merging:
  1. Start with B_k for smallest k in either queue.
  2. If only one B_k, add B_k to new queue and go to next k.
  3. Merge two B_k's to get new B_{k+1} by making larger root the child of smaller root. Go to step 2 with k = k + 1.

Binomial Queues: Merge Example

- Merge H1 and H2

H1: H2:



Binomial Queues: Merge and Insert

- What is the run time for Merge of two O(N) queues?
- How would you insert a new item into the queue?

Binomial Queues: Merge and Insert

- What is the run time for Merge of two O(N) queues?
  - O(number of trees) = O(log N)
- How would you insert a new item into the queue?
  - Create a single node queue B_0 with new item and merge with existing queue
  - Again, O(log N) time
- On-board example: Insert 1, 2, 3, …,7 into an empty binomial queue

Binomial Queues: DeleteMin

- Steps:
  1. Find tree B_k with the smallest root
  2. Remove B_k from the queue
  3. Delete root of B_k (return this value); You now have a second queue made up of the forest B_0, B_1, …, B_{k-1}
  4. Merge this queue with remainder of the original (from step 2)
- Example: Insert 1, 2, …, 7 into empty queue and DeleteMin
- Run time analysis: Steps 1 through 4 = how much time for an N-node queue?

Binomial Queues: DeleteMin

- Steps:
  1. Find tree B_k with the smallest root
  2. Remove B_k from the queue
  3. Delete root of B_k (return this value); You now have a new queue made up of the forest B_0, B_1, …, B_{k-1}
  4. Merge this queue with remainder of the original (from step 2)
- Example: Insert 1, 2, …, 7 into empty queue and DeleteMin
- Run time analysis: Step 1 is O(log N), step 2 and 3 are O(1), and step 4 is O(log N). Total time = O(log N)

Implementation of Binomial Queues

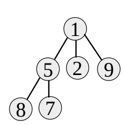
- DeleteMin requires fast access to all subtrees of root
  - Need pointer-based implementation
  - Use First-Child/Next-Sibling representation of trees
- Merge adds one binomial tree as child to another
  - This added tree will now be the largest subtree
- Question: Should we order subtrees in increasing or decreasing size?

Implementation of Binomial Queues

- DeleteMin requires fast access to all subtrees of root
  - Need pointer-based implementation
  - Use First-Child/Next-Sibling representation of trees
  - Use array of pointers to root nodes of binomial trees
- Merge adds one binomial tree as child to another
  - This added tree will now be the largest subtree
- Question: Should we order subtrees in increasing or decreasing size?
  - Order in terms of decreasing subtree size
  - Avoids traversal of linked list of next sibling pointers
- What does our queue containing 1, 2, …, 7 look like?

Other Priority Queues: d-Heaps

- Similar to a binary heap, except we allow more than 2 children per node
- d-heap has d children per node
- Example: 3-heap – root is A[1]; children of node A[i] are at what locations?



Other Priority Queues: d-Heaps

- Similar to a binary heap, except we allow more than 2 children per node
- d-heap has d children per node
- Example: 3-heap – root is A[1] and children of node A[i] are A[3i-1], A[3i], A[3i+1]
- Just as in B-tree, more children means shallower heap
  - Depth is $O(\log_d N)$ instead of $O(\log_2 N)$
  - But, d-1 comparisons to find smallest child
  - Tradeoff between depth and "breadth"
  - Optimal d value is application dependent