# CSE 332 Winter 2024
# Lecture 18: Graphs

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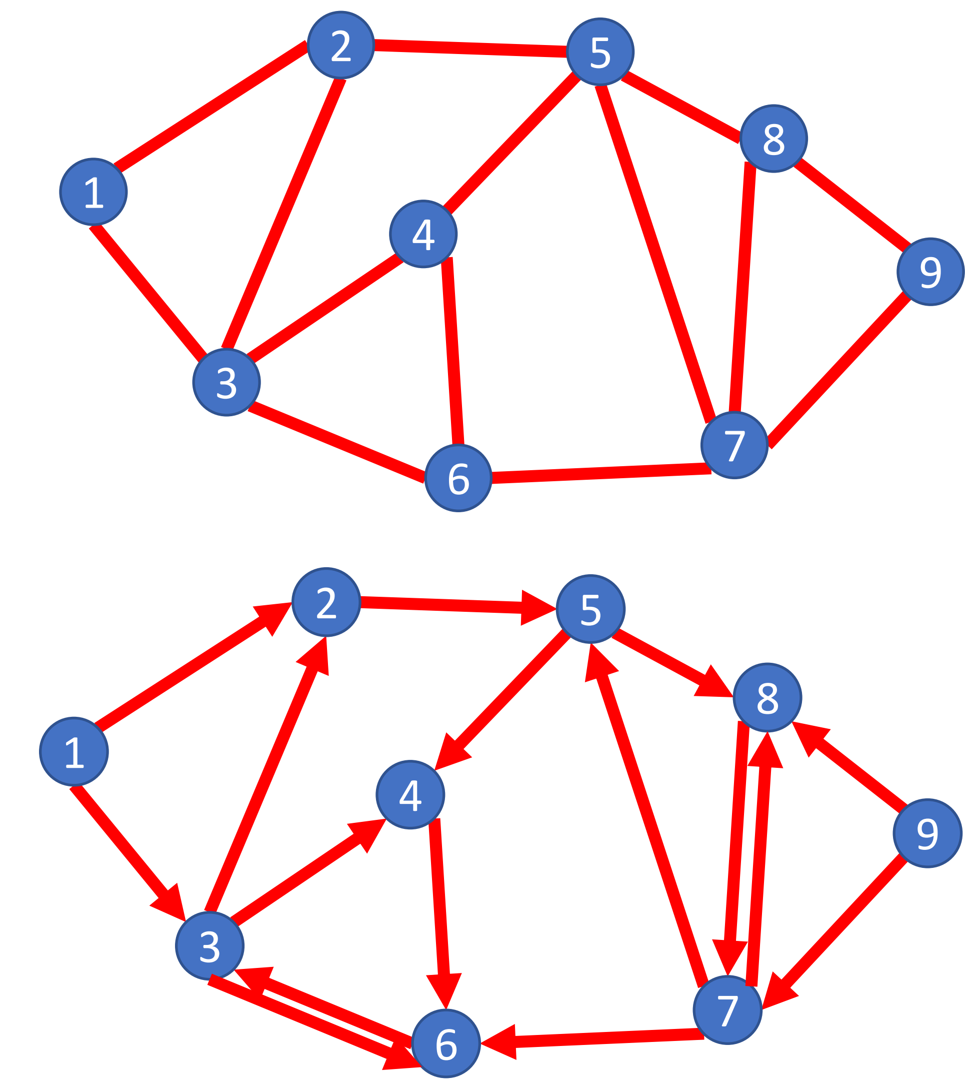
# Some Graph Terms

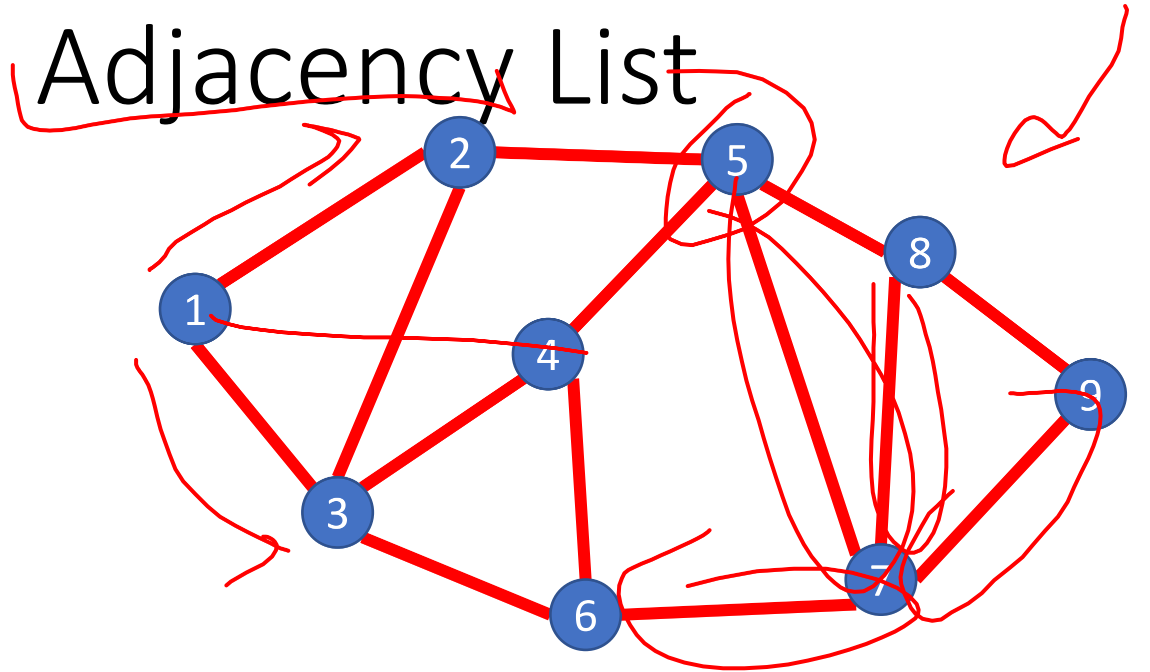
- Adjacent/Neighbors
  - Nodes are adjacent/neighbors if they share an edge
- Degree
  - Number of “neighbors” of a vertex
- Indegree
  - Number of incoming neighbors
- Outdegree
  - Number of outgoing neighbors



# Graph Operations

- To represent a Graph (i.e. build a data structure) we need:
  - Add Edge
  - Remove Edge
  - Check if Edge Exists
  - Get Neighbors (incoming)
  - Get Neighbors (outgoing)

# Adjacency List




| | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | | |
| 2 | 1 | 3 | 5 | |
| 3 | 1 | 2 | 4 | 6 |
| 4 | 3 | 5 | 6 | |
| 5 | 2 | 4 | 7 | 8 |
| 6 | 3 | 4 | 7 | |
| 7 | 5 | 6 | 8 | 9 |
| 8 | 5 | 7 | 9 | |
| 9 | 7 | 8 | | |

<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n + m)$

Add Edge: $\Theta(1)$

Remove Edge: $\Theta(\deg(v))$

Check if Edge Exists: $\Theta(\deg(v))$

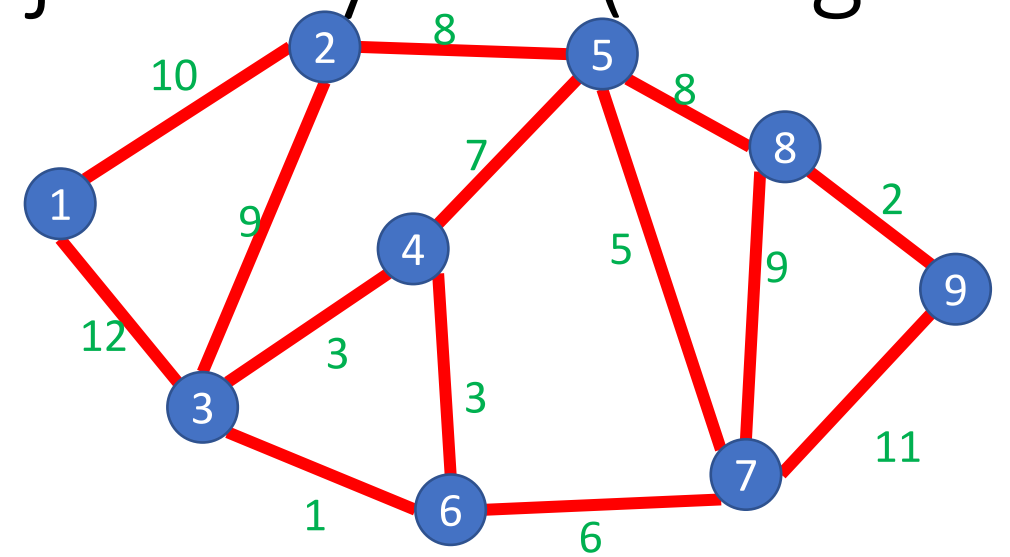
Get Neighbors (incoming): $\Theta(n + m)$

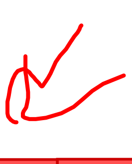
Get Neighbors (outgoing): $\Theta(\deg(v))$

$|V| = n$

$|E| = m$

# Adjacency List (Weighted)



| Vertex | | | | |
|---|---|---|---|---|
| 1 | 2 | 3 | | |
| 2 | 1 | 3 | 5 | |
| 3 | 1 | 2 | 4 | 6 |
| 4 | 3 | 5 | 6 | |
| 5 | 2 | 4 | 7 | 8 |
| 6 | 3 | 4 | 7 | |
| 7 | 5 | 6 | 8 | 9 |
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| 9 | 7 | 8 | | |

<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n+m)$

Add Edge: $\Theta(1)$

Remove Edge: $\Theta(\deg(v))$

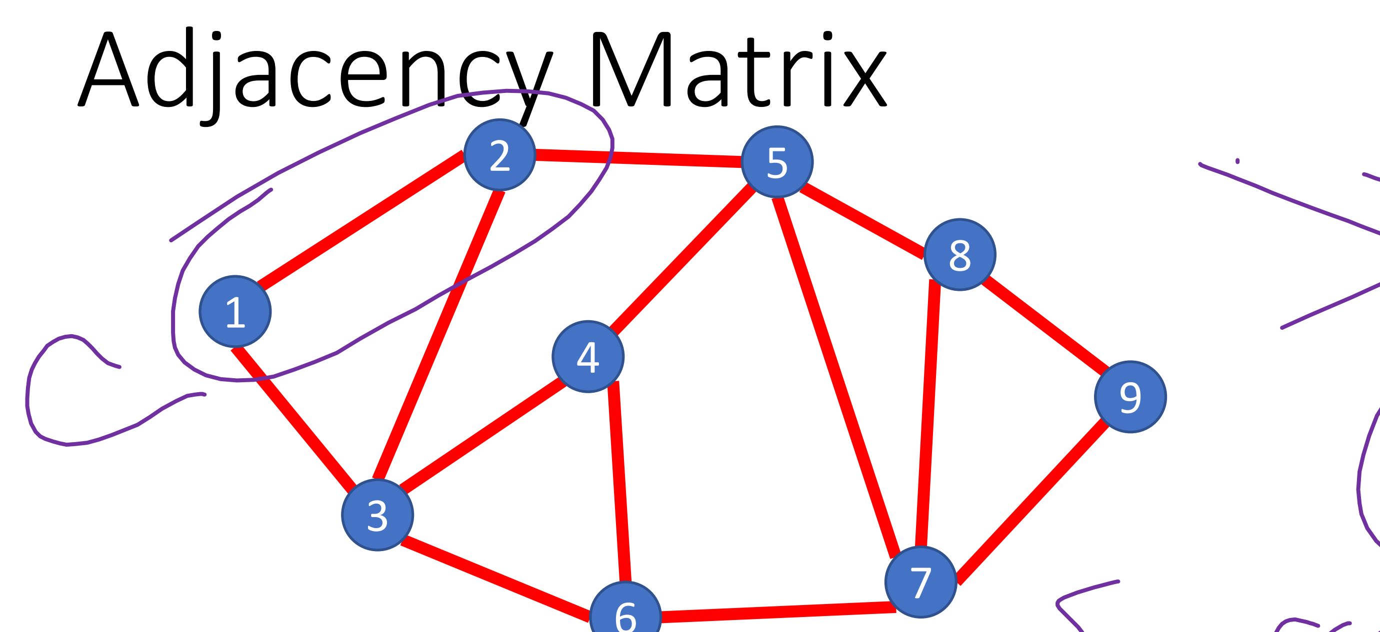
Check if Edge Exists: $\Theta(\deg(v))$

Get Neighbors (incoming): $\Theta(n+m)$

Get Neighbors (outgoing): $\Theta(\deg(v))$

$|V| = n$
$|E| = m$

# Adjacency Matrix




| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| A | | 1 | 1 | | | | | | |
| B | 1 | | 1 | | 1 | | | | |
| C | 1 | 1 | | 1 | | 1 | | | |
| D | | | 1 | | 1 | 1 | | | |
| E | | 1 | | 1 | | | 1 | 1 | |
| F | | | 1 | 1 | | | 1 | | |
| G | | | | | 1 | 1 | | 1 | 1 |
| H | | | | | 1 | | 1 | | 1 |
| I | | | | | | | 1 | 1 | |

Time/Space Tradeoffs

Space to represent: $\Theta(?)$

Add Edge: $\Theta(?)$

Remove Edge: $\Theta(?)$

Check if Edge Exists: $\Theta(?)$

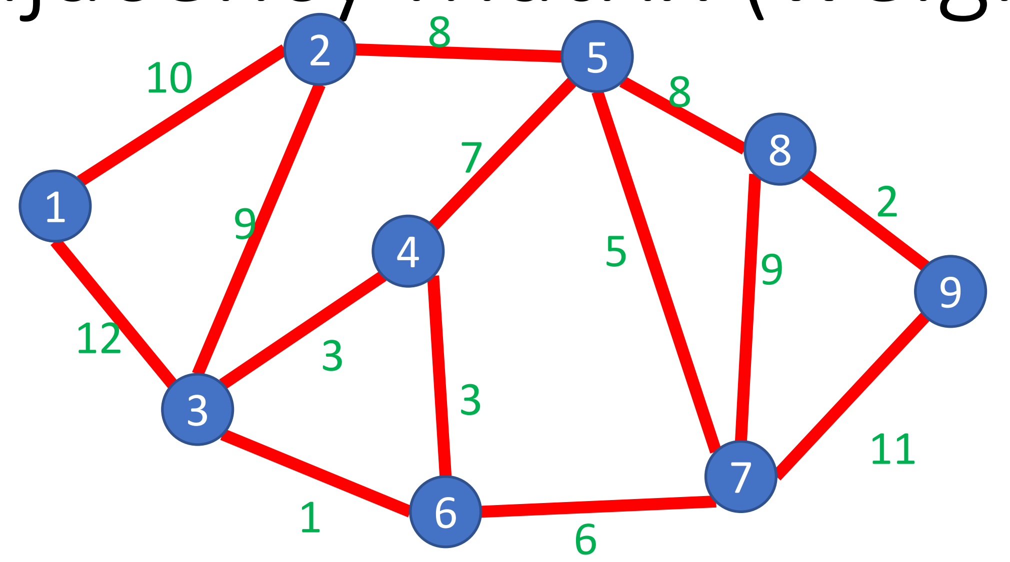
Get Neighbors (incoming): $\Theta(?)$

Get Neighbors (outgoing): $\Theta(?)$

$|V| = n$

$|E| = m$

# Adjacency Matrix (weighted)




| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| A | | 1 | 1 | | | | | | |
| B | 1 | | 1 | | 1 | | | | |
| C | 1 | 1 | | 1 | | 1 | | | |
| D | | | 1 | | 1 | 1 | | | |
| E | | 1 | | 1 | | | 1 | 1 | |
| F | | | 1 | 1 | | | 1 | | |
| G | | | | | 1 | 1 | | 1 | 1 |
| H | | | | | 1 | | 1 | | 1 |
| I | | | | | | | 1 | 1 | |

<u>Time/Space Tradeoffs</u>

Space to represent: $\Theta(n^2)$

Add Edge: $\Theta(1)$

Remove Edge: $\Theta(1)$

Check if Edge Exists: $\Theta(1)$

Get Neighbors (incoming): $\Theta(n)$

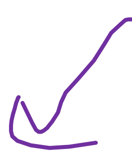
Get Neighbors (outgoing): $\Theta(n)$

$|V| = n$
$|E| = m$

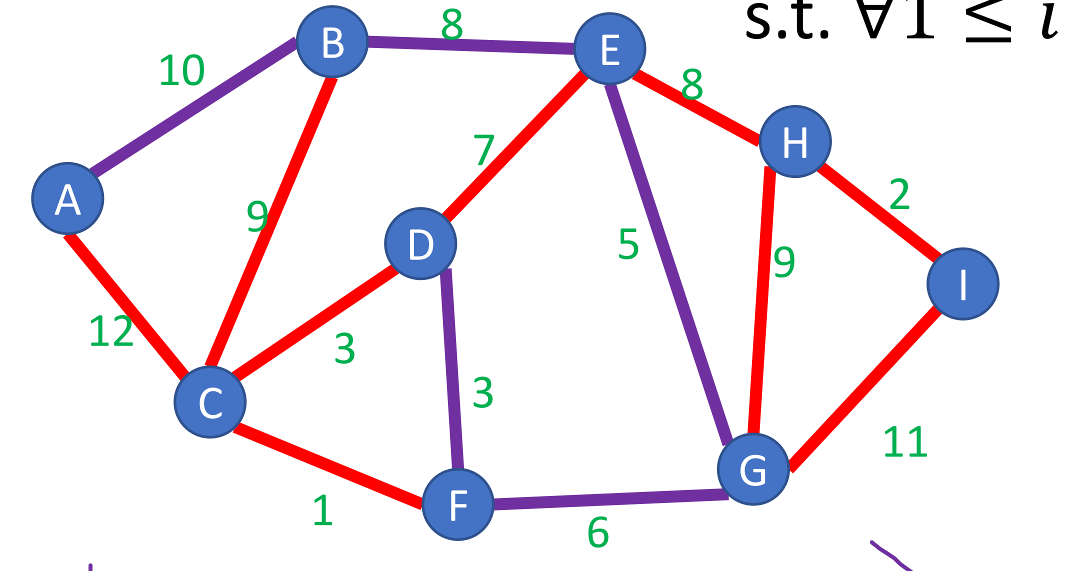
# Aside

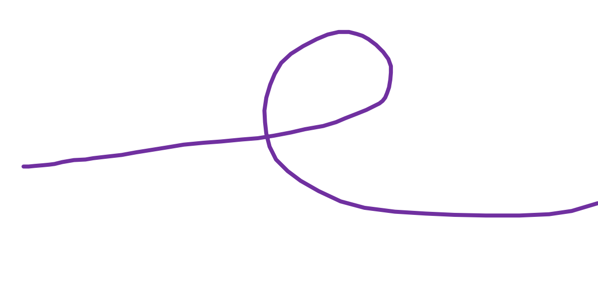
- Almost always, adjacency lists are the better choice
- Most graphs are missing most of their edges, so the adjacency list is much more space efficient and the slower operations aren't that bad

# Definition: Path



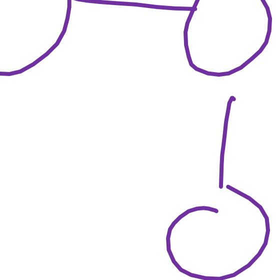
A sequence of nodes $(v_1, v_2, \dots, v_k)$
s.t. $\forall 1 \leq i \leq k-1,\ (v_i, v_{i+1}) \in E$



Simple Path:
A path in which each node appears at most once

Cycle:
A path which starts and ends in the same place



# Definition: (Strongly) Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from $v_1$ to $v_2$

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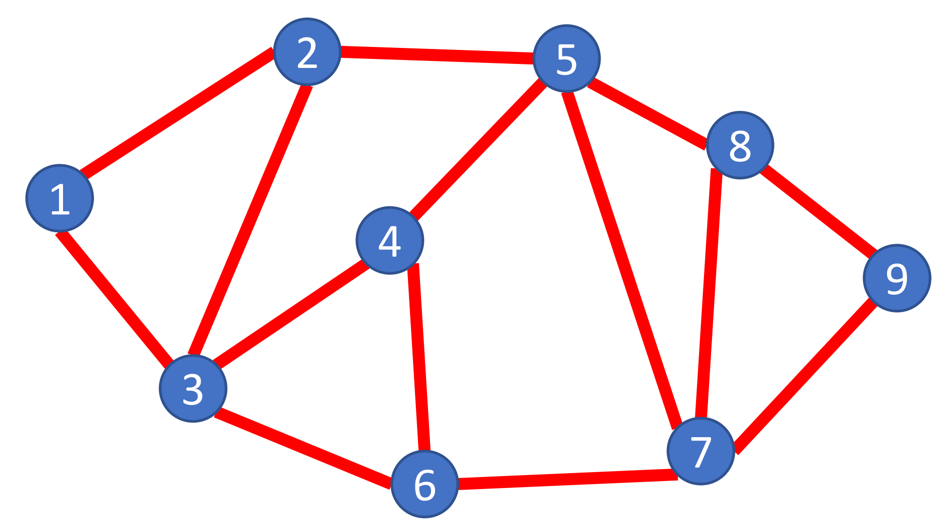
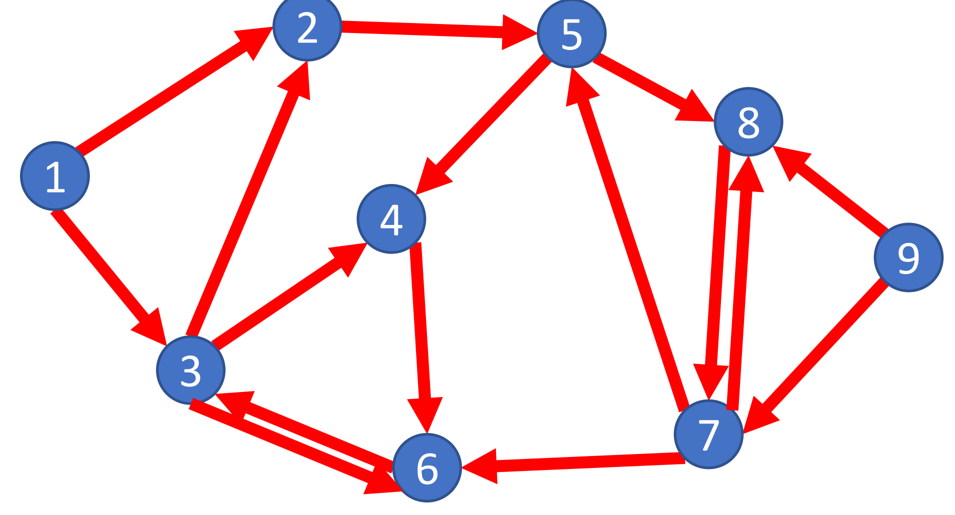


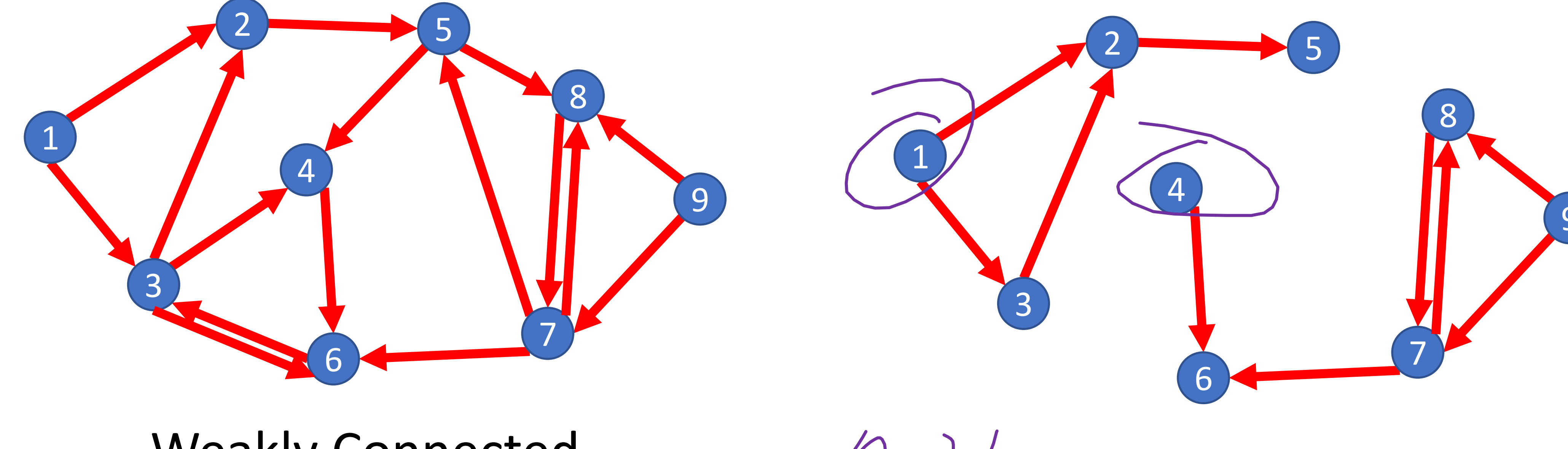

Connected




Not (strongly) Connected

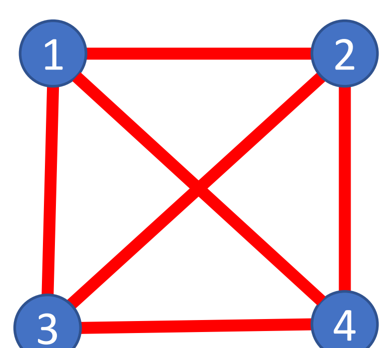
# Definition: Weakly Connected Graph

A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is a path from $v_1$ to $v_2$ ignoring direction of edges

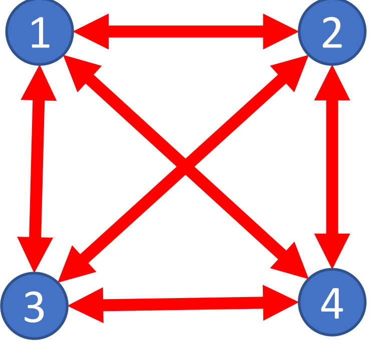


# Definition: Complete Graph

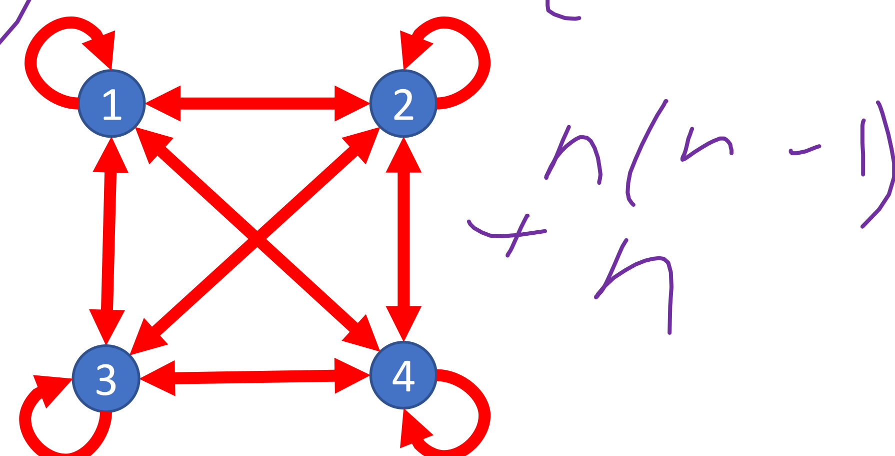
A Graph $G = (V, E)$ s.t. for any pair of nodes $v_1, v_2 \in V$ there is an edge from $v_1$ to $v_2$




Complete Undirected Graph




Complete Directed Graph



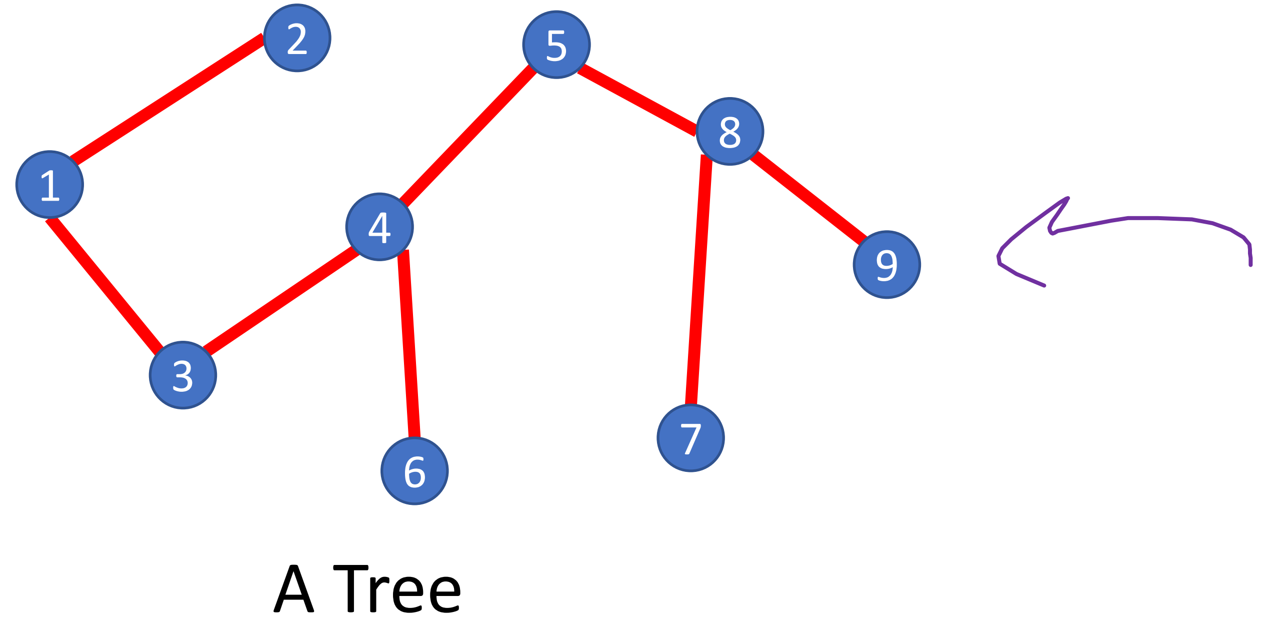

Complete Directed Non-simple Graph

# Graph Density, Data Structures, Efficiency

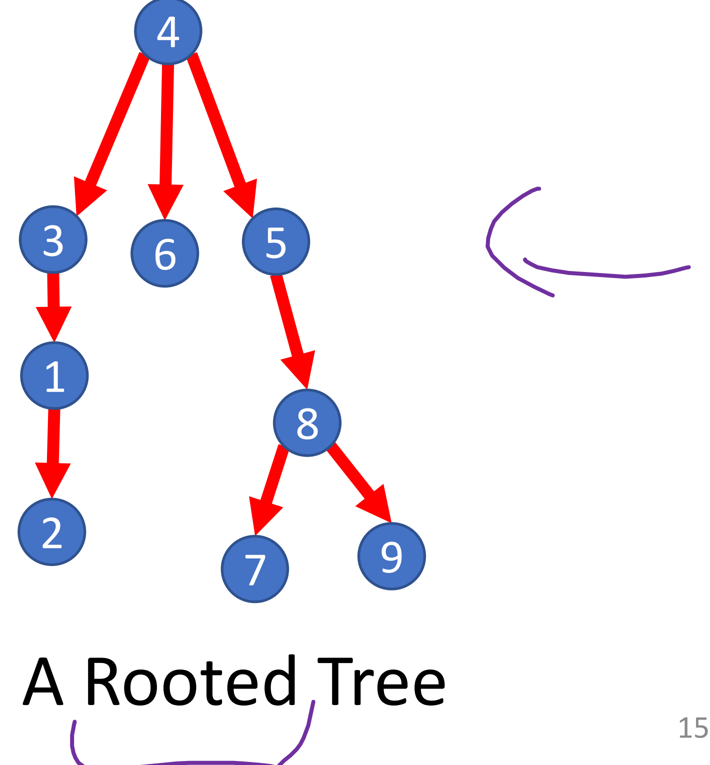
- The maximum number of edges in a graph is $\Theta(|V|^2)$:
  - Undirected and simple: $\frac{|V|(|V|-1)}{2}$
  - Directed and simple: $|V|(|V|-1)$
  - Direct and non-simple (but no duplicates): $|V|^2$
- If the graph is connected, the minimum number of edges is $|V|-1$
- If $|E| \in \Theta(|V|^2)$ we say the graph is **dense**
- If $|E| \in \Theta(|V|)$ we say the graph is **sparse**
- Because $|E|$ is not always near to $|V|^2$ we do not typically substitute $|V|^2$ for $|E|$ in running times, but leave it as a separate variable

# Definition: Tree

A Graph $G = (V, E)$ is a tree if it is undirect, connected, and has no cycles (i.e. is acyclic).
Often one node is identified as the “root”



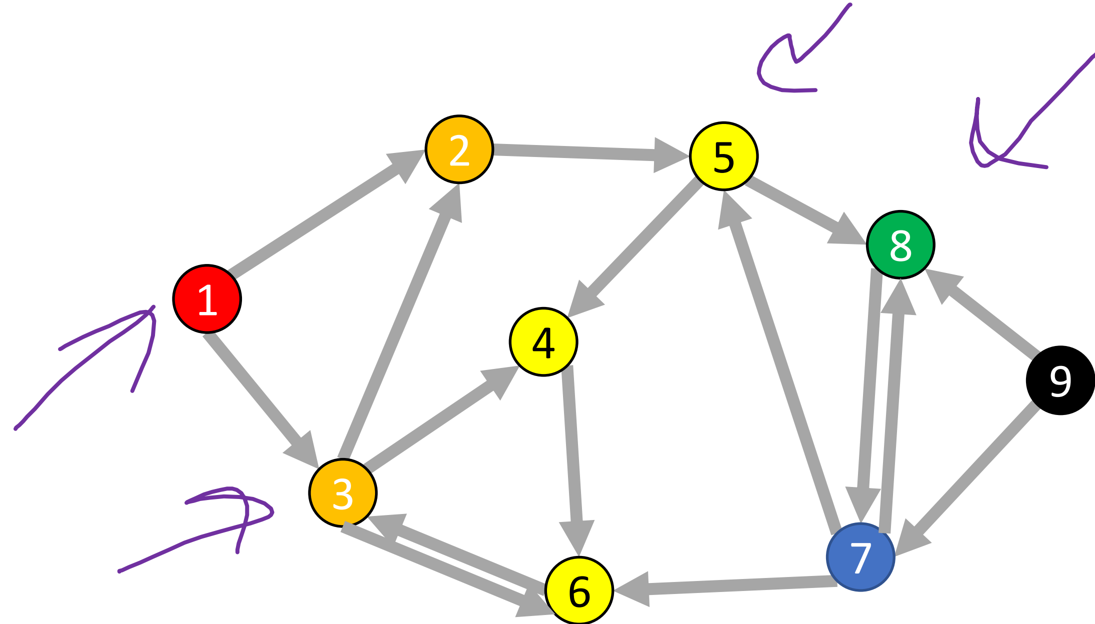

A Tree




A Rooted Tree

# Breadth-First Search

- Input: a node $s$
- Behavior: Start with node $s$, visit all neighbors of $s$, then all neighbors of neighbors of $s$, …
- Output:
  - How long is the shortest path?
  - Is the graph connected?

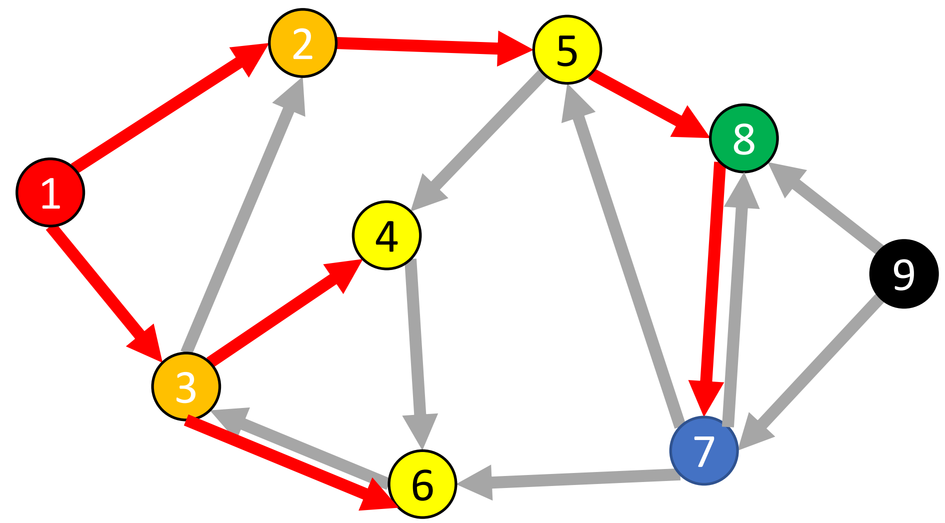


# BFS

```
void bfs(graph, s){
        found = new Queue();
        found.enqueue(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.dequeue();
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                found.enqueue(v);
                        }
                }
        }
}
```

Running time: $\Theta(|V| + |E|)$

# Shortest Path (unweighted)



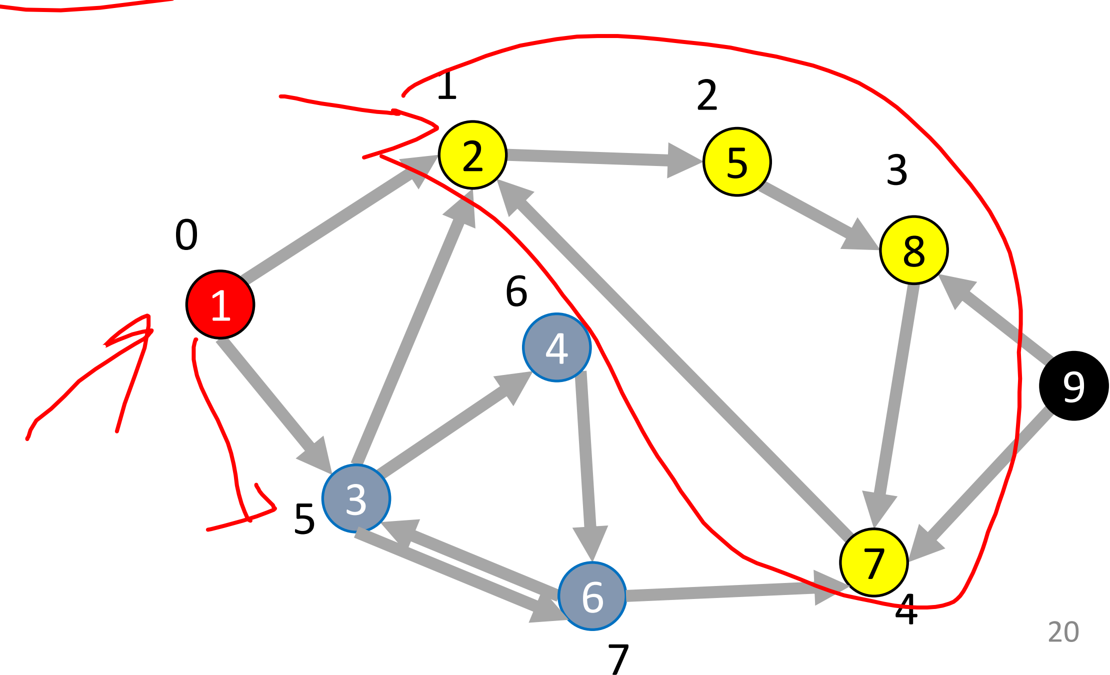

Idea: when it's seen, remember its "layer" depth!

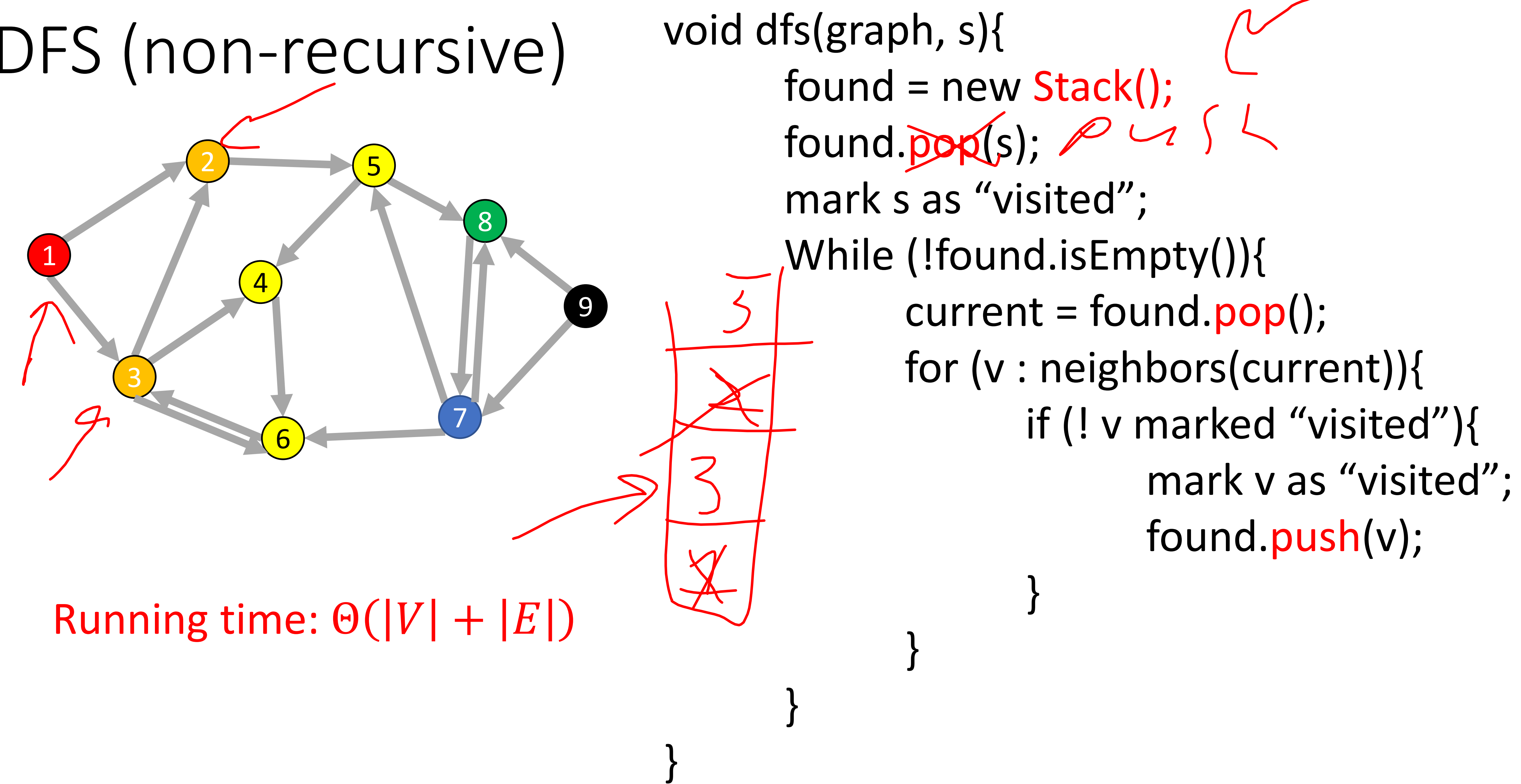
```
int shortestPath(graph, s, t){
    found = new Queue();
    layer = 0;
    found.enqueue(s);
    mark s as "visited";
    While (!found.isEmpty()){
        current = found.dequeue();
        layer = depth of current;
        for (v : neighbors(current)){
            if (! v marked "visited"){
                mark v as "visited";
                depth of v = layer + 1;
                found.enqueue(v);
            }
        }
    }
    return depth of t;
}
```

# Depth-First Search

# Depth-First Search

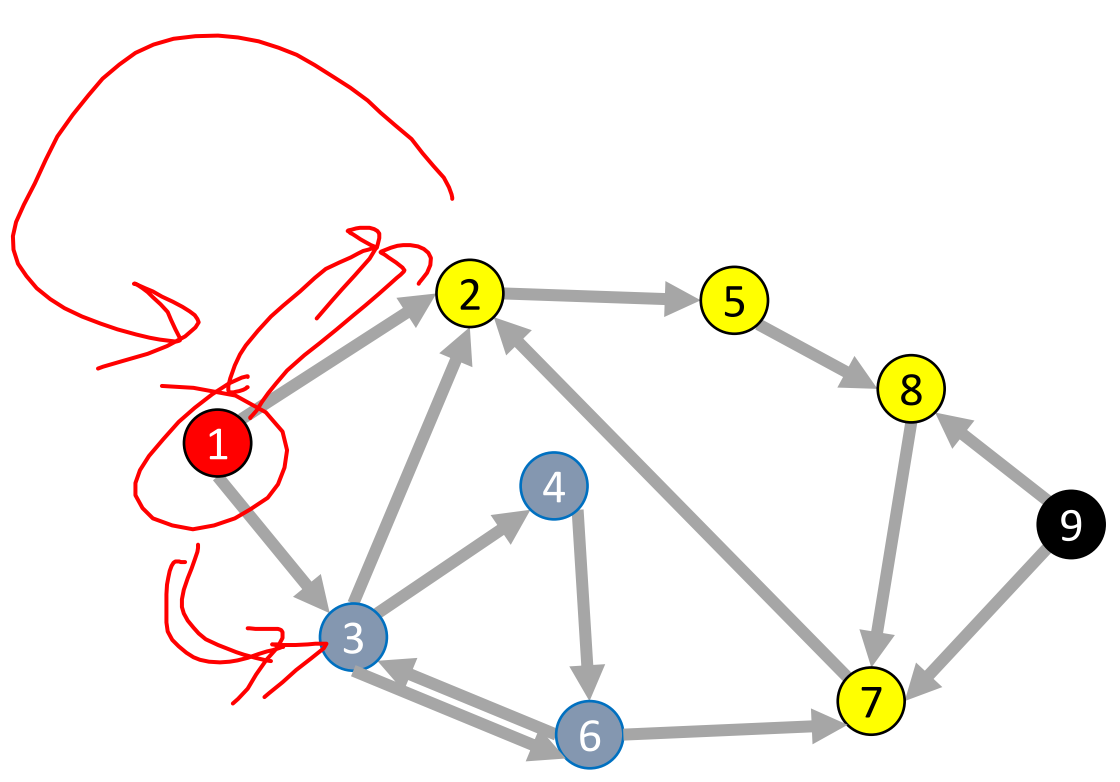
- Input: a node $s$
- Behavior: Start with node $s$, visit one neighbor of $s$, then all nodes reachable from that neighbor of $s$, then another neighbor of $s$,...
  - Before moving on to the second neighbor of $s$, visit everything reachable from the first neighbor of $s$
- Output:
  - Does the graph have a cycle?
  - A **topological sort** of the graph.




DFS (non-recursive)
1
2
3
4
5
6
7
8
9
void dfs(graph, s){
found = new Stack();
found.pop(s);
push
mark s as “visited”;
While (!found.isEmpty()){
current = found.pop();
for (v : neighbors(current)){
if (! v marked “visited”){
mark v as “visited”;
found.push(v);
}
}
}
}
5
3
Running time: Θ(|V| + |E|)

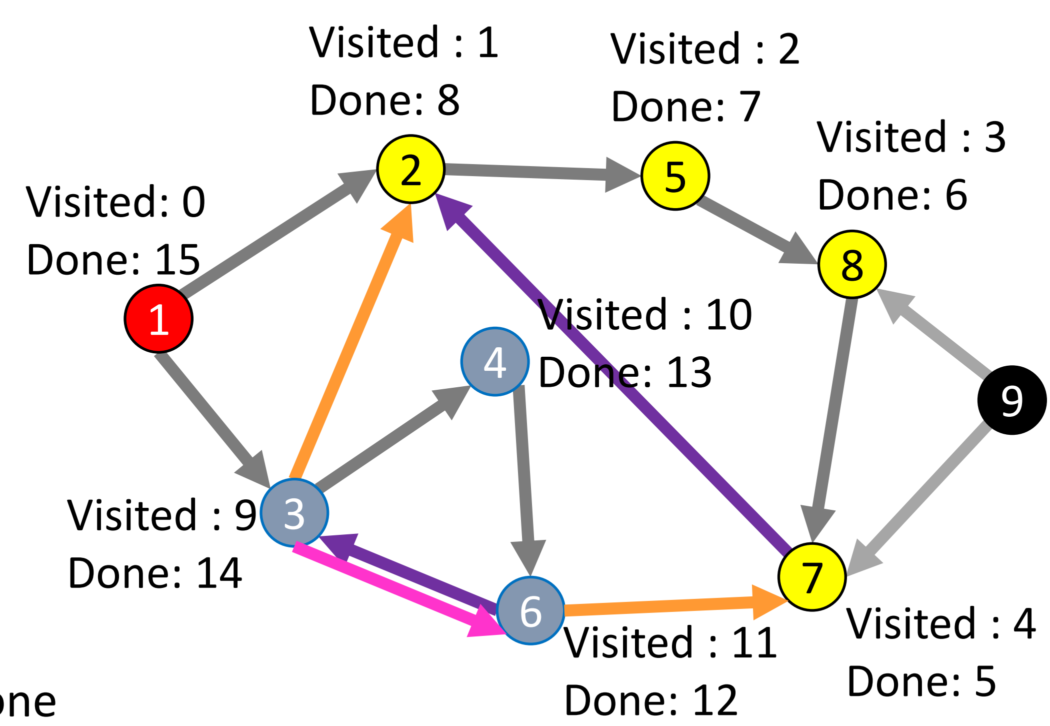
# DFS Recursively (more common)

```
void dfs(graph, curr){
        mark curr as "visited";
        for (v : neighbors(current)){
                if (! v marked "visited"){
                        dfs(graph, v);
                }
        }
        mark curr as "done";
}
```



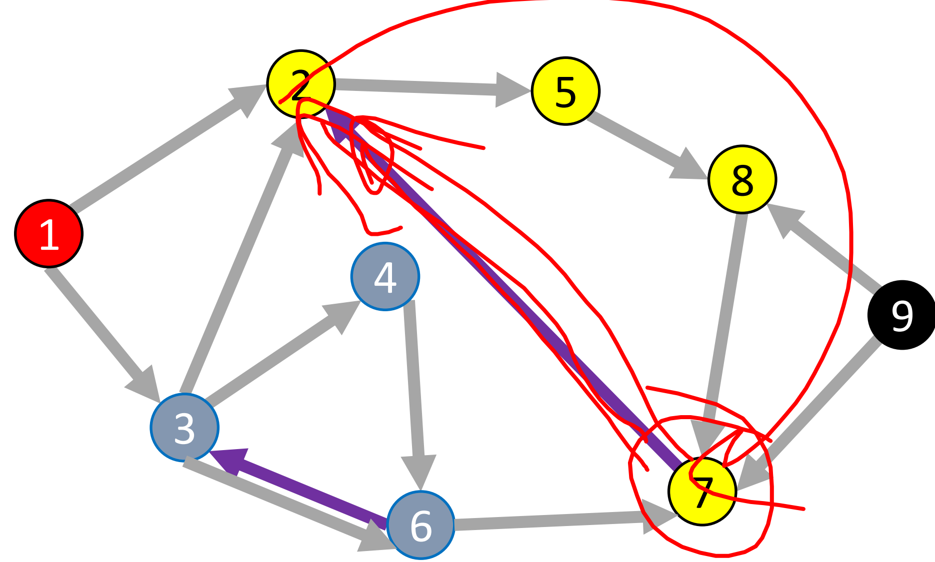
# Using DFS

- Consider the "visited times" and "done times"
- Edges can be categorized:
  - Tree Edge
    - $(a, b)$ was followed when pushing
    - $(a, b)$ when $b$ was unvisited when we were at $a$
  - Back Edge
    - $(a, b)$ goes to an "ancestor"
    - $a$ and $b$ visited but not done when we saw $(a, b)$
    - $t_{visited}(b) < t_{visited}(a) < t_{done}(a) < t_{done}(b)$
  - Forward Edge
    - $(a, b)$ goes to a "descendent"
    - $b$ was visited and done between when $a$ was visited and done
    - $t_{visited}(a) < t_{visited}(b) < t_{done}(b) < t_{done}(a)$
  - Cross Edge
    - $(a, b)$ goes to a node that doesn't connect to $a$
    - $b$ was seen and done before $a$ was ever visited
    - $t_{done}(b) < t_{visited}(a)$



# Back Edges

- Behavior of DFS:
  - “Visit everything reachable from the current node before going back”
- Back Edge:
  - The current node’s neighbor is an “in progress” node
  - Since that other node is “in progress”, the current node is reachable from it
  - The back edge is a path to that other node
  - **Cycle!**



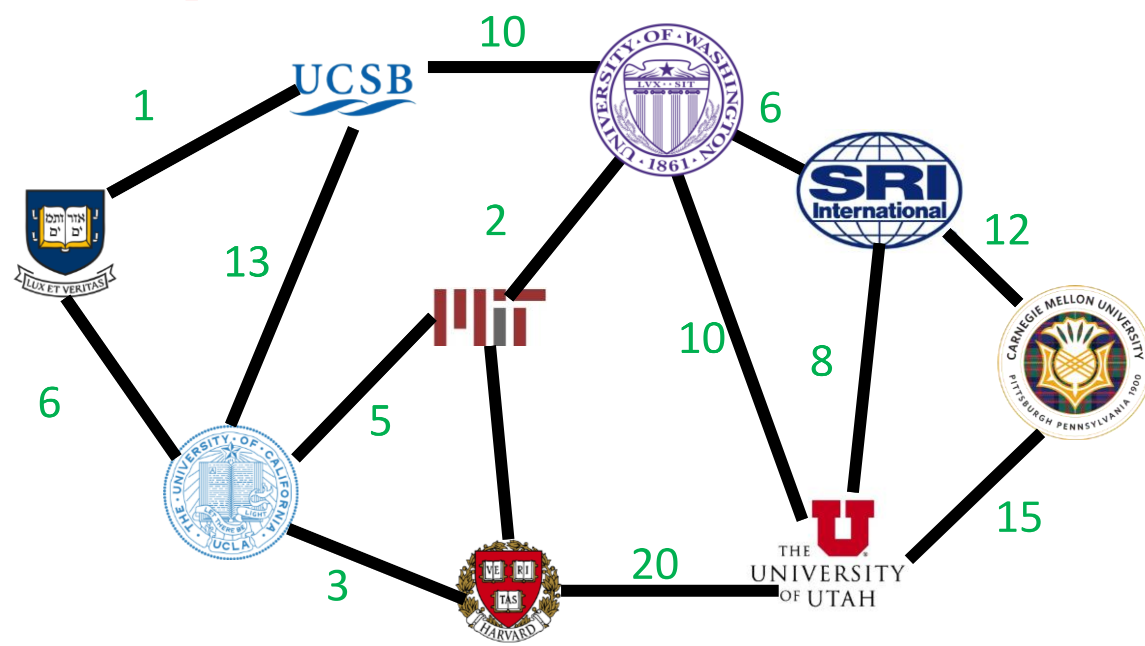
# Cycle Detection

Idea: Look for a back edge!

```
boolean hasCycle(graph, curr){
        mark curr as "visited";
        cycleFound = false;
        for (v : neighbors(current)){
                if (v marked "visited" && ! v marked "done"){
                        cycleFound=true;
                }
                if (! v marked "visited" && !cycleFound){
                        cycleFound = hasCycle(graph, v);
                }
        }
        mark curr as "done";
        return cycleFound;
}
```

1
2
3
4
5
6
7
8
9

# Single-Source Shortest Path



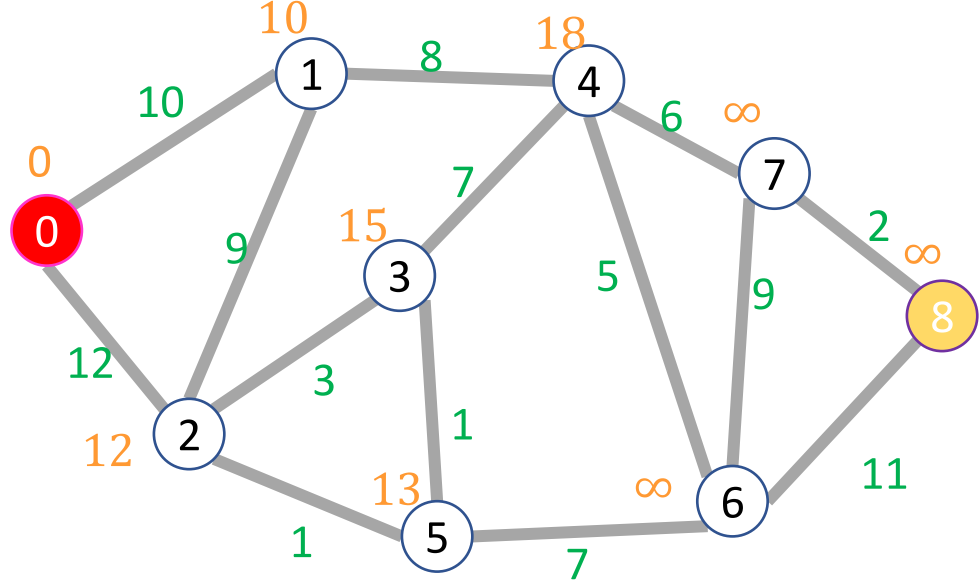

Find the quickest way to get from UVA to each of these other places

Given a graph $G = (V, E)$ and a start node $s \in V$, for each $v \in V$ find the least-weight path from $s \rightarrow v$ (call this weight $\delta(s, v)$)

(assumption: all edge weights are positive)

# Dijkstra's Algorithm

- Input: graph with **no negative edge weights**, start node $s$, end node $t$
- Behavior: Start with node $s$, repeatedly go to the incomplete node "nearest" to $s$, stop when
- Output:
  - Distance from start to end
  - Distance from start to every node



# Dijkstra’s Algorithm

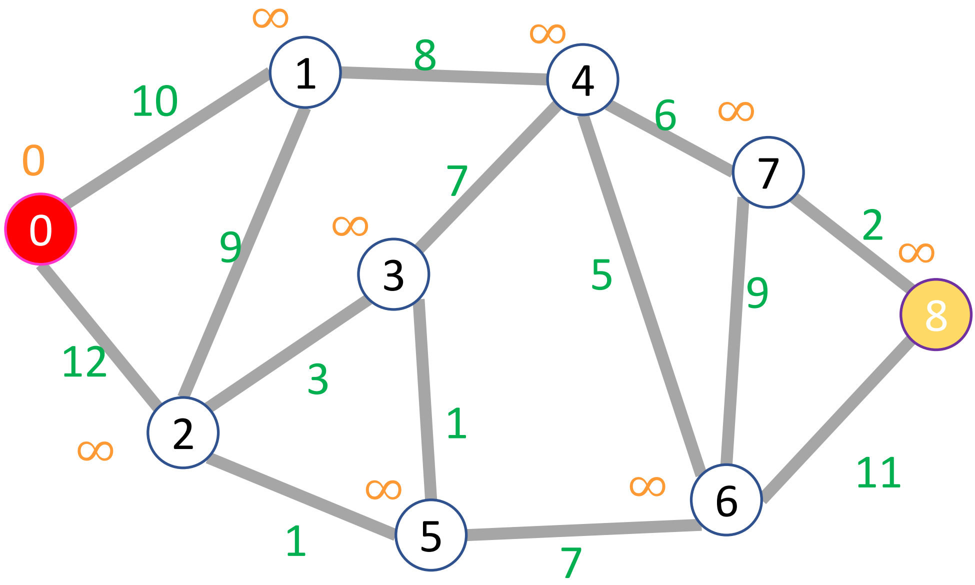
Start: 0

End: 8

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path

| Node | Done? |
|---|---|
| 0 | F |
| 1 | F |
| 2 | F |
| 3 | F |
| 4 | F |
| 5 | F |
| 6 | F |
| 7 | F |
| 8 | F |

| Node | Distance |
|---|---|
| 0 | 0 |
| 1 | ∞ |
| 2 | ∞ |
| 3 | ∞ |
| 4 | ∞ |
| 5 | ∞ |
| 6 | ∞ |
| 7 | ∞ |
| 8 | ∞ |



# Dijkstra’s Algorithm

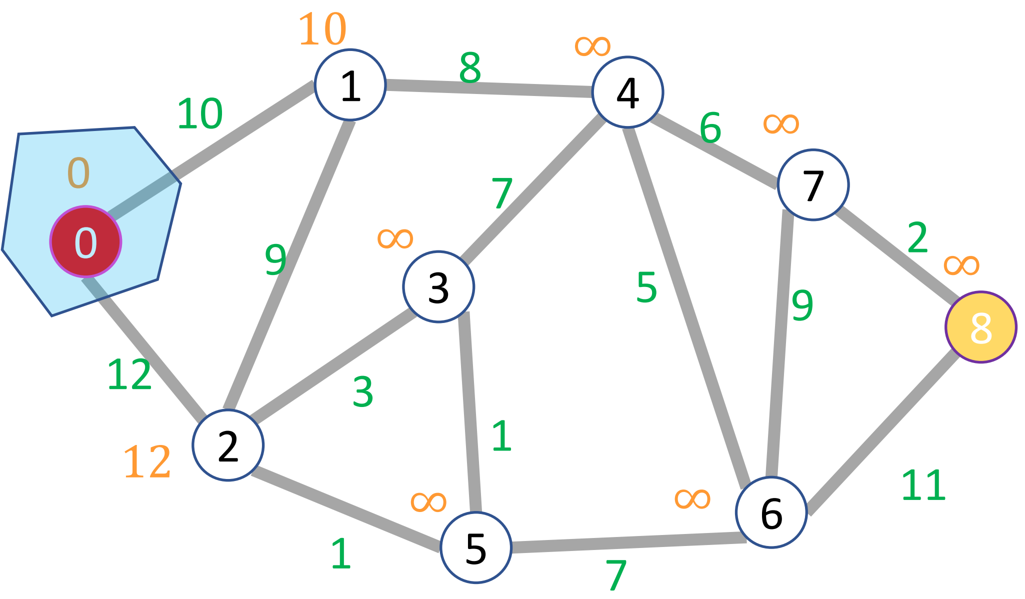
Start: 0

End: 8

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path

| Node | Done? |
| --- | --- |
| 0 | T |
| 1 | F |
| 2 | F |
| 3 | F |
| 4 | F |
| 5 | F |
| 6 | F |
| 7 | F |
| 8 | F |

| Node | Distance |
| --- | --- |
| 0 | 0 |
| 1 | 10 |
| 2 | 12 |
| 3 | ∞ |
| 4 | ∞ |
| 5 | ∞ |
| 6 | ∞ |
| 7 | ∞ |
| 8 | ∞ |



# Dijkstra’s Algorithm

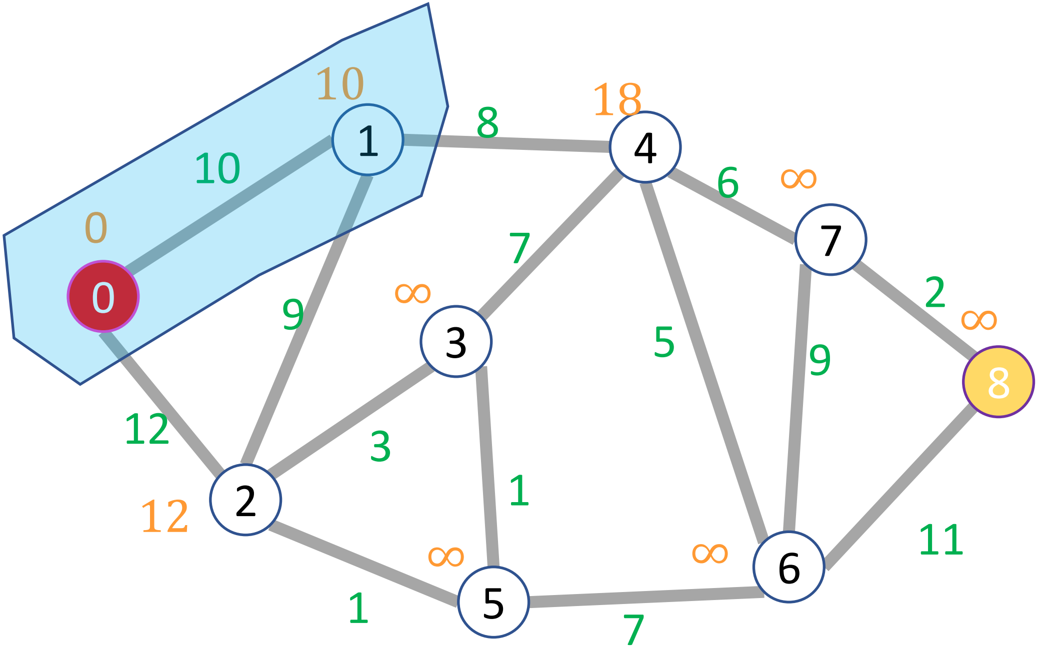
Start: 0

End: 8

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path

| Node | Done? |
|---|---|
| 0 | T |
| 1 | T |
| 2 | F |
| 3 | F |
| 4 | F |
| 5 | F |
| 6 | F |
| 7 | F |
| 8 | F |

| Node | Distance |
|---|---|
| 0 | 0 |
| 1 | 10 |
| 2 | 12 |
| 3 | ∞ |
| 4 | 18 |
| 5 | ∞ |
| 6 | ∞ |
| 7 | ∞ |
| 8 | ∞ |



# Dijkstra's Algorithm

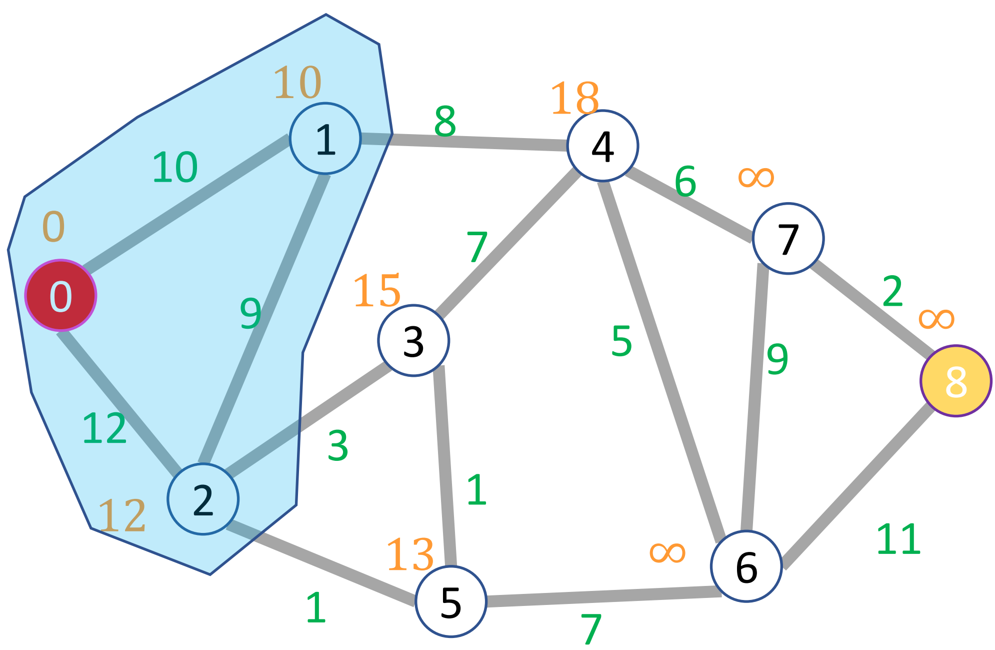
Start: 0

End: 8

| Node | Done? |
|---|---|
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | F |
| 4 | F |
| 5 | F |
| 6 | F |
| 7 | F |
| 8 | F |

| Node | Distance |
|---|---|
| 0 | 0 |
| 1 | 10 |
| 2 | 12 |
| 3 | 15 |
| 4 | 18 |
| 5 | 13 |
| 6 | ∞ |
| 7 | ∞ |
| 8 | ∞ |

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path



# Dijkstra’s Algorithm

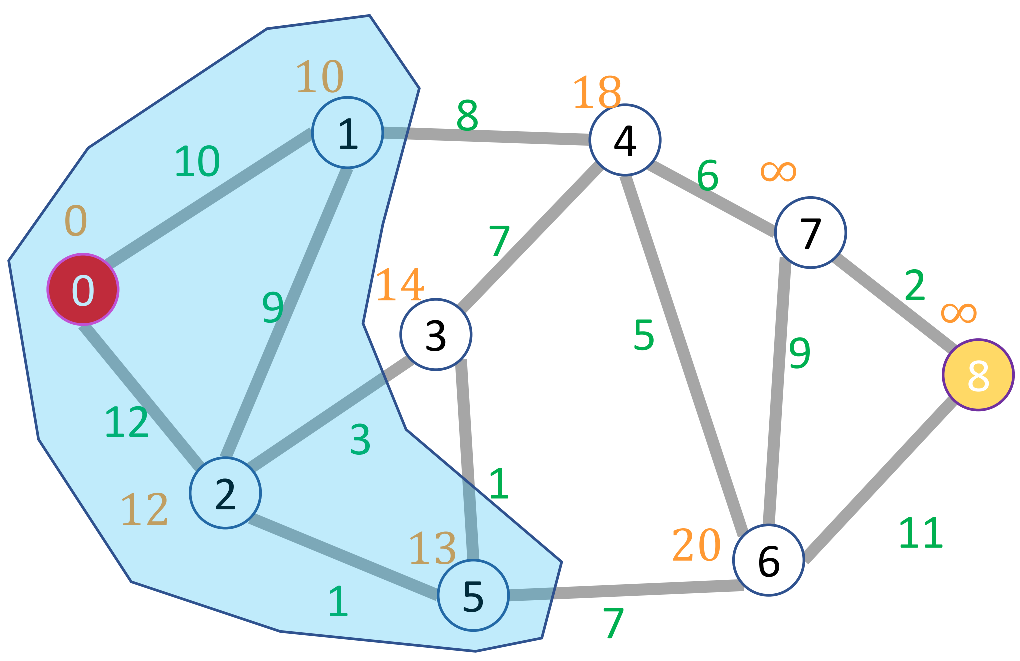
Start: 0

End: 8

Idea: When a node is the closest undiscovered thing to the start, we have found its shortest path

| Node | Done? |
| --- | --- |
| 0 | T |
| 1 | T |
| 2 | T |
| 3 | F |
| 4 | F |
| 5 | T |
| 6 | F |
| 7 | F |
| 8 | F |

| Node | Distance |
| --- | --- |
| 0 | 0 |
| 1 | 10 |
| 2 | 12 |
| 3 | 14 |
| 4 | 18 |
| 5 | 13 |
| 6 | ∞ |
| 7 | 20 |
| 8 | ∞ |



# Dijkstra's Algorithm

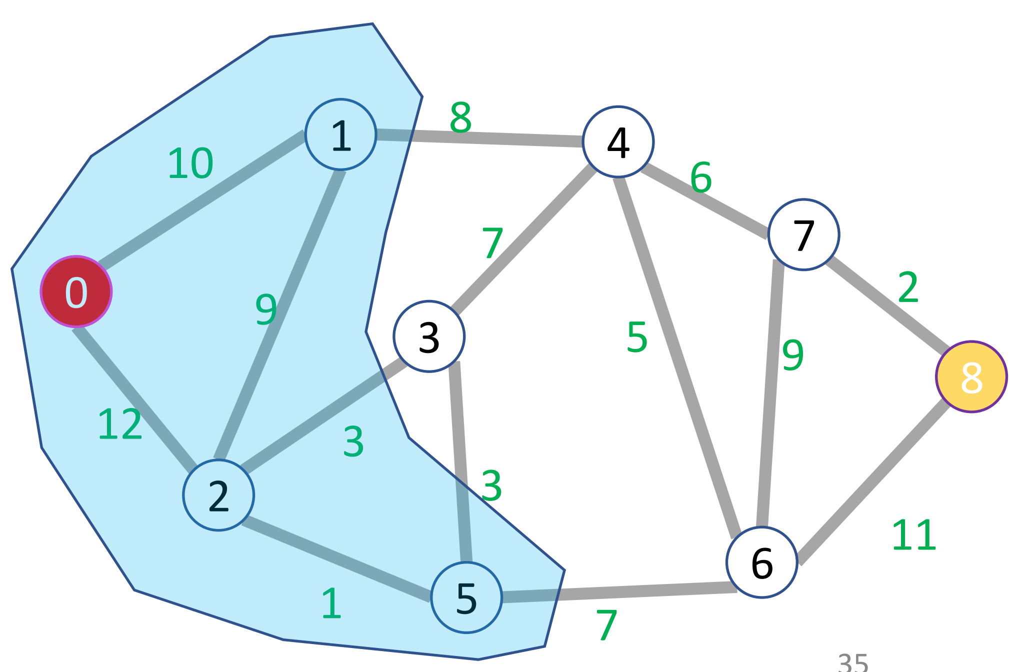
```
int dijkstras(graph, start, end){
        distances = [∞, ∞, ∞,…];  // one index per node
        done = [False,False,False,…];  // one index per node
        PQ = new minheap();
        PQ.insert(0, start);  // priority=0, value=start
        distances[start] = 0;
        while (!PQ.isEmpty){
                current = PQ.deleteMin();
                done[current] = true;
                for (neighbor : current.neighbors){
                        if (!done[neighbor]){
                                new_dist = distances[current]+weight(current,neighbor);
                                if new_dist < distances[neighbor]{
                                        distances[neighbor] = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return distances[end]
}
```

# Dijkstra's Algorithm: Running Time

- How many total priority queue operations are necessary?
  - How many times is each node added to the priority queue?
  - How many times might a node's priority be changed?
- What's the running time of each priority queue operation?
- Overall running time:
  - $\Theta(|E| \log|V|)$

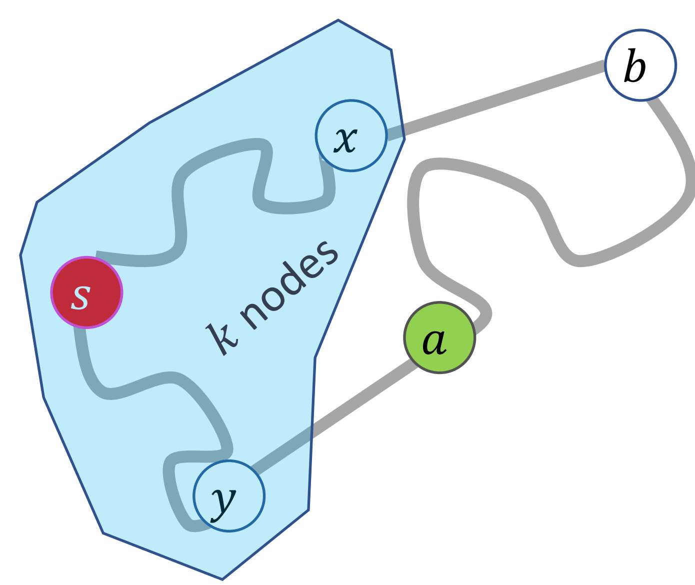
# Dijkstra's Algorithm: Correctness

- Claim: when a node is removed from the priority queue, we have found its shortest path
- Induction over number of completed nodes
- Base Case:
- Inductive Step:



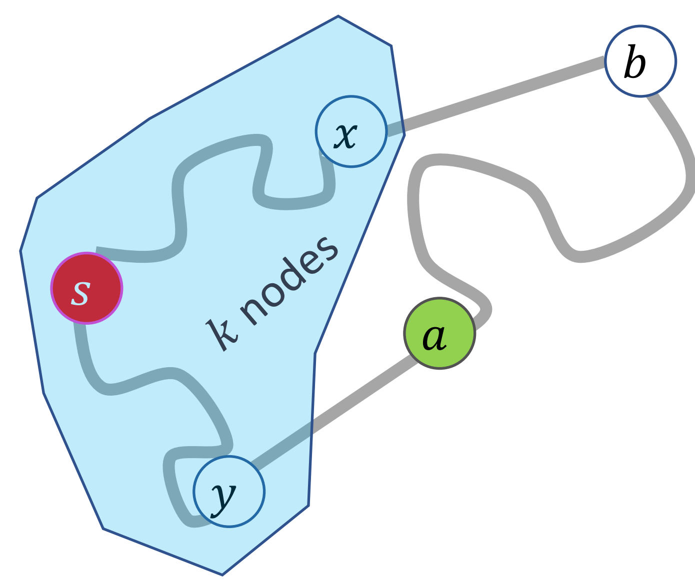
# Dijkstra's Algorithm: Correctness

- Claim: when a node is removed from the priority queue, its distance is that of the shortest path
- Induction over number of completed nodes
- Base Case: Only the start node removed
  - It is indeed 0 away from itself
- Inductive Step:
  - If we have correctly found shortest paths for the first $k$ nodes, then when we remove node $k + 1$ we have found its shortest path



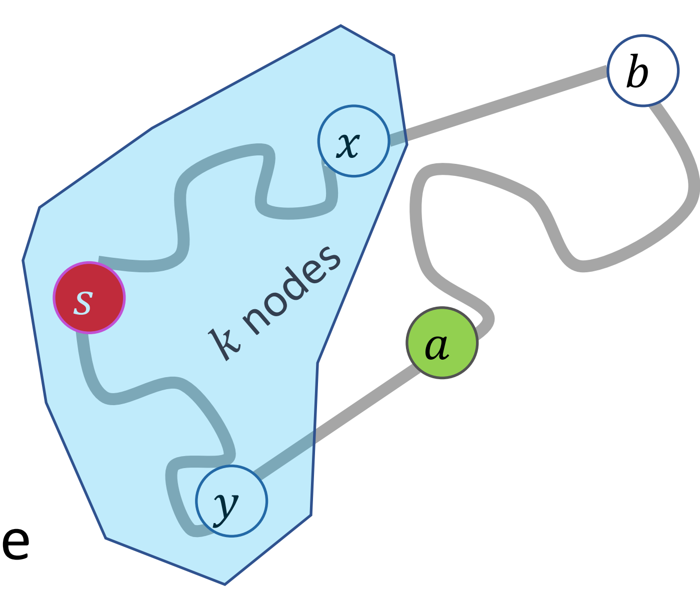
# Dijkstra's Algorithm: Correctness

- Suppose $a$ is the next node removed from the queue. What do we know bout $a$?



# Dijkstra's Algorithm: Correctness

- Suppose $a$ is the next node removed from the queue.
  - No other node incomplete node has a shorter path discovered so far
- Claim: no undiscovered path to $a$ could be shorter
  - Consider any other incomplete node $b$ that is 1 edge away from a complete node
  - $a$ is the closest node that is one away from a complete node
  - Thus no path that includes $b$ can be a shorter path to $a$
  - Therefore the shortest path to $a$ must use only complete nodes, and therefore we have found it already!



# Dijkstra's Algorithm: Correctness

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  - No other node incomplete node has a shorter path discovered so far
- Claim: no undiscovered path to $a$ could be shorter
  - Consider any other incomplete node $b$ that is 1 edge away from a complete node
  - $a$ is the closest node that is one away from a complete node
  - No path from $b$ to $a$ can have negative weight
  - Thus no path that includes $b$ can be a shorter path to $a$
  - Therefore the shortest path to $a$ must use only complete nodes, and therefore we have found it already!