CSE 332 Winter 2024
Lecture 14: Sorting

Nathan Brunelle

http://www.cs.uw.edu/332

Sorting

- Rearrangement of items into some defined sequence
  - Usually: reordering a list from smallest to largest according to some metric
- Why sort things?
  - Enables binary search
  - Human readability
  - Sorting is a helpful preprocessing step for other algorithms

More Formal Definition

- Input:
  - An array A of items
  - A comparison function for these items
    - Given two items x and y, we can determine whether $x < y$, $x > y$, or $x = y$
- Output:
  - A permutation of A such that if $i \leq j$ then $A[i] \leq A[j]$
  - Permutation: a sequence of the same items but perhaps in a different order

Sorting “Landscape”

- There is no singular best algorithm for sorting
- Some are faster, some are slower
- Some use more memory, some use less
- Some are super extra fast if your data meets certain assumptions
- Some have other special properties that make them valuable
- No sorting algorithm can have only all the “best” attributes

Properties to consider

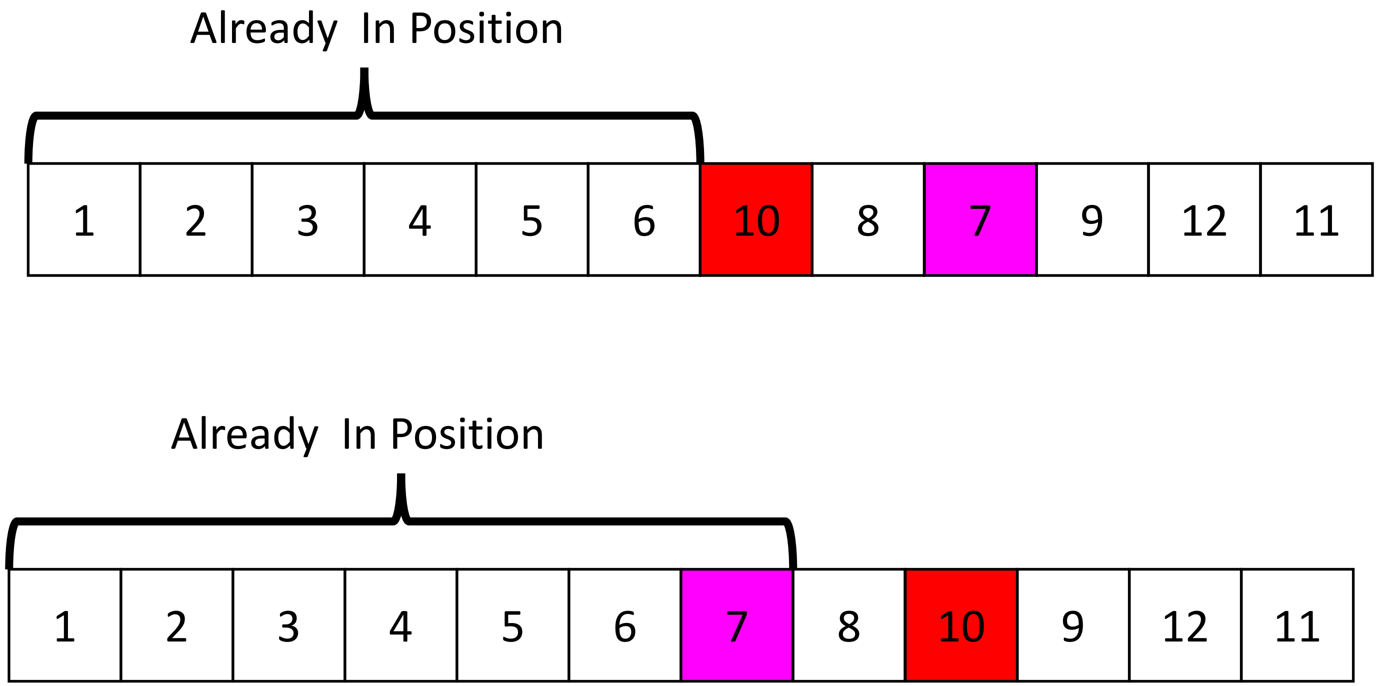
- Running time
  - What is the worst case running time?
  - What is the best case?
  - Does the algorithm run faster if the list is close to sorted?
    - If so, we call it Adaptive
- Memory Usage
  - How much memory does the algorithm use in addition to the array?
    - If $\Theta(1)$ then we call it In-Place
      - Sorts things by only swapping things in the same array we started with.
- What happens when there is a "tie"?
  - If "tied" elements are guaranteed to remain in the same relative order, this is called a Stable Sort
  - E.g. a stable sort guarantees that, after sorting by the first initial and then by last initial, "N.J.B." will come before "S.C.B"

“In Place” Sorting Algorithm

- A sorting algorithm which requires no extra data structures
- Idea: It sorts items just by swapping things in the same array given
- Definition: it only uses $\Theta(1)$ extra space

Selection Sort

- Idea: Find the next smallest element, swap it into the next index in the array



Selection Sort

- Swap the thing at index 0 with the smallest thing in the array
- Swap the thing at index 1 with the smallest thing after index 0
- …
- Swap the thing at index i with the smallest thing after index $i-1$

```
for (i=0; i<a.length; i++){
    smallest = i;
    for (j=i; j<a.length; j++){
        if (a[j]<a[smallest]){ smallest=j;}
    }
    temp = a[i];
    a[i] = a[smallest];
    a[smallest] = a[i];
}
```

Running Time:

Worst Case: $\Theta(\cdot)$

Best Case: $\Theta(\cdot)$

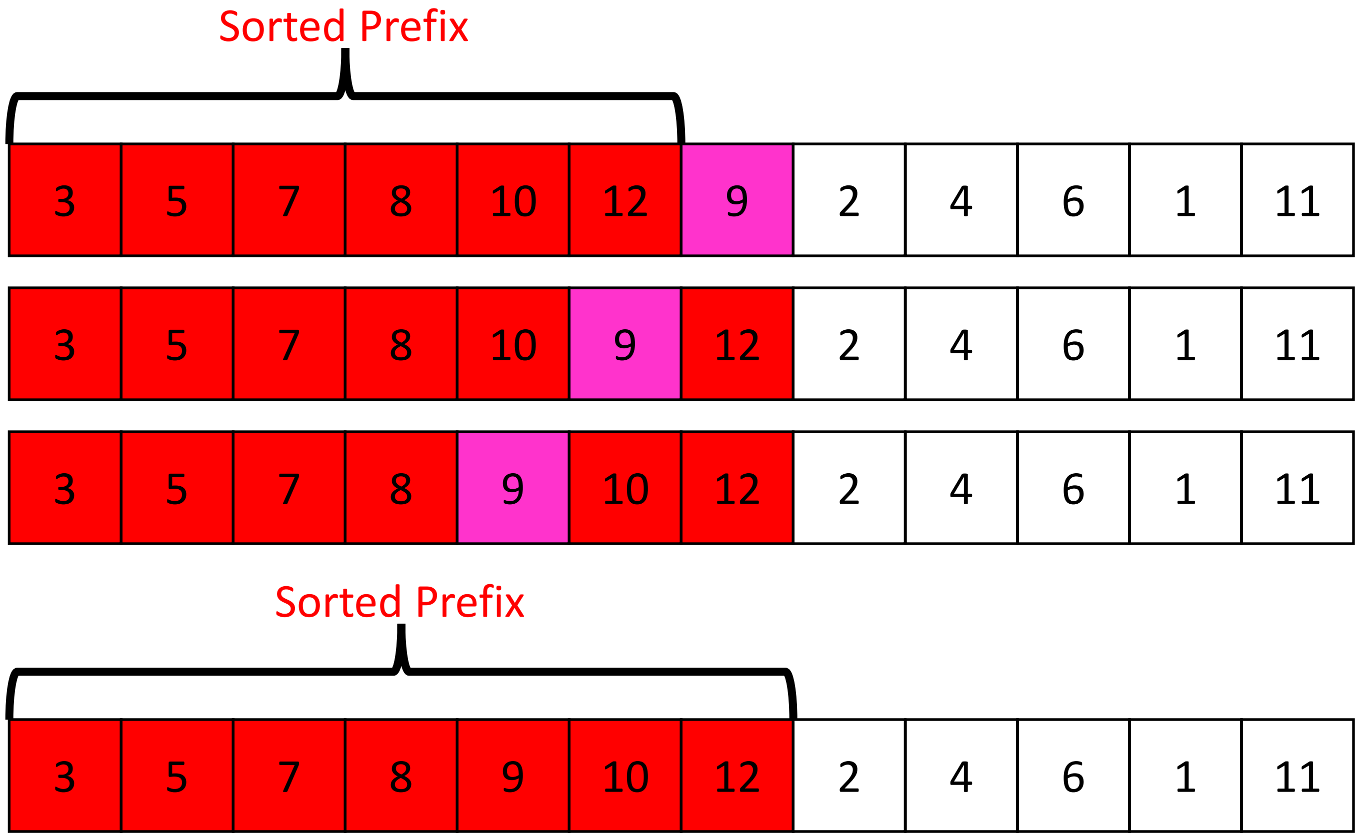
10	77	5	15	2	22	64	41	18	19	30	21	3	24	23	33
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Properties of Selection Sort

- In-Place?
  - Yes!
- Adaptive?
  - No
- Stable?
  - Yes!
  - As long as you always pick the left-most element when there’s a “tie”

Insertion Sort

- Idea: Maintain a sorted list prefix, extend that prefix by "inserting" the next element



Insertion Sort

- If the items at index 0 and 1 are out of order, swap them
- Keep swapping the item at index 2 with the thing to its left as long as the left thing is larger
- …
- Keep swapping the item at index i with the thing to its left as long as the left thing is larger

```
for (i=1; i<a.length; i++){
    prev = i-1;
    while(a[i] < a[prev] && prev > -1){
        temp = a[i];
        a[i] = a[prev];
        a[prev] = a[i];
        i--;
        prev--;
    }
}
```

Running Time:

Worst Case: $\Theta(\cdot)$

Best Case: $\Theta(\cdot)$

10	77	5	15	2	22	64	41	18	19	30	21	3	24	23	33
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Properties of Insertion Sort

- In-Place?
  - Yes!
- Adaptive?
  - Yes!
- Stable?
  - Yes!
  - As long as you don't swap when there's a tie
- Online!
  - You can begin sorting the list before you have all the elements
  - "Insert" items as they arrive

Aside: Bubble Sort – we won’t cover it

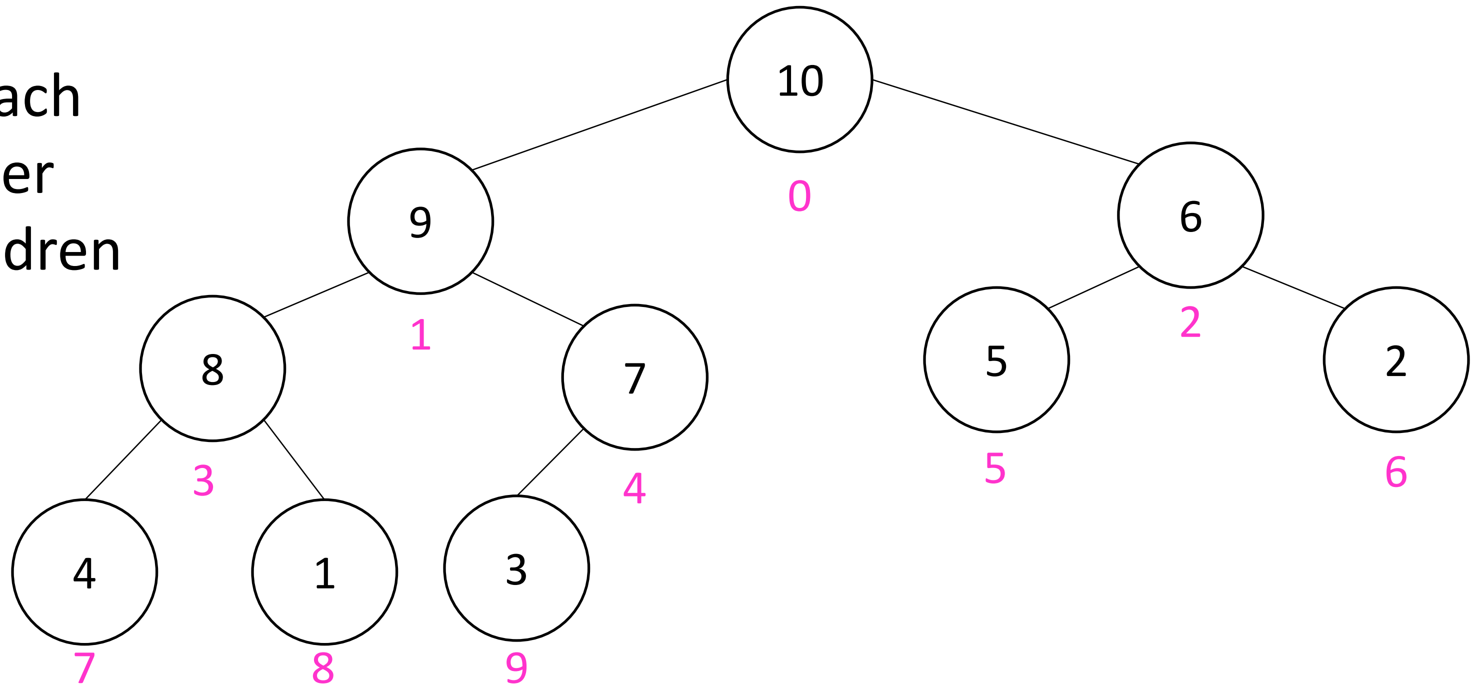
"the bubble sort seems to have nothing to recommend it, except a catchy name and the fact that it leads to some interesting theoretical problems” –Donald Knuth, The Art of Computer Programming

Heap Sort

- Idea: Build a maxHeap, repeatedly delete the max element from the heap to build sorted list Right-to-Left

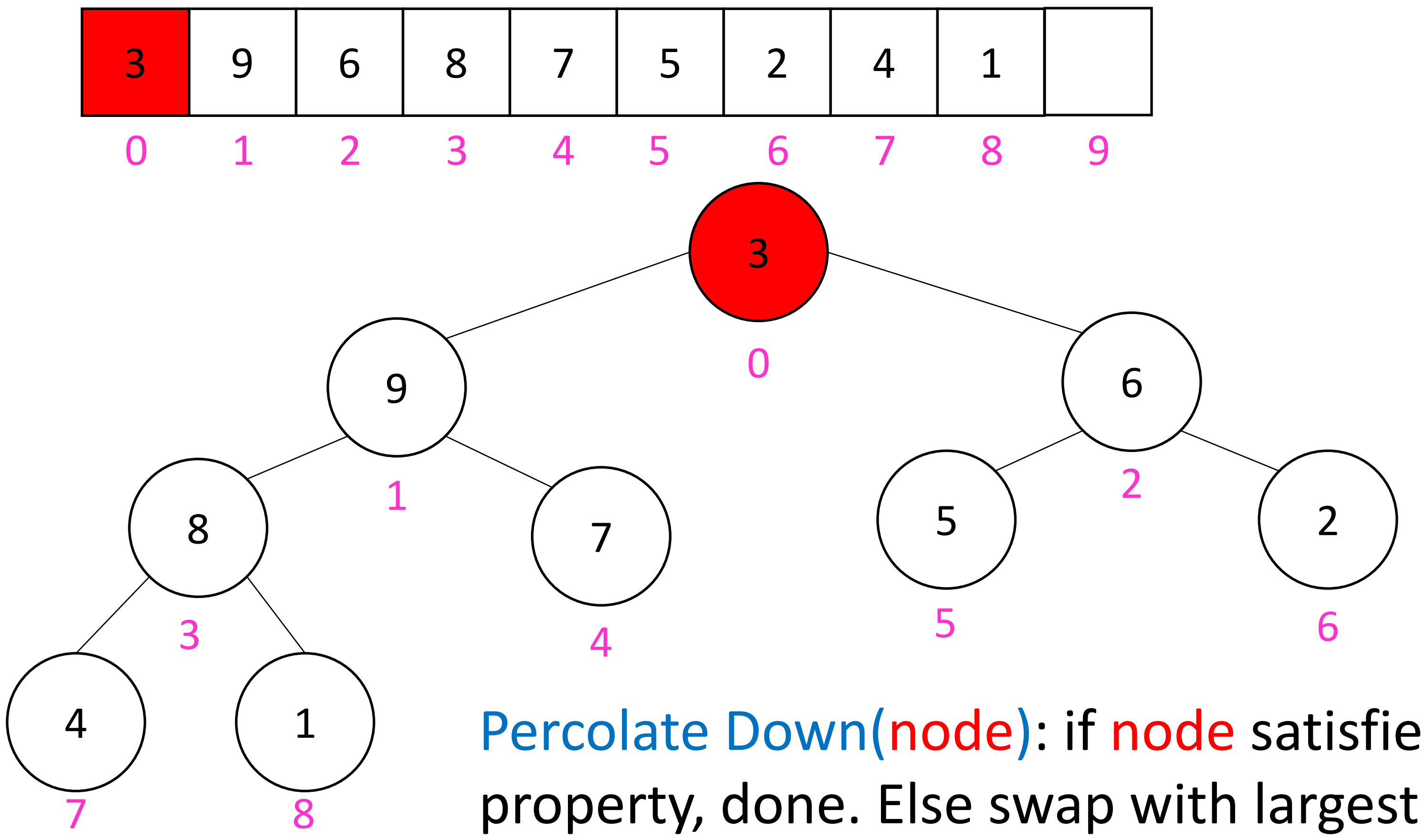
10	9	6	8	7	5	2	4	1	3
0	1	2	3	4	5	6	7	8	9

Max Heap Property: Each node is larger than its children



Heap Sort

- Remove the Max element (i.e. the root) from the Heap: replace with last element, call percolateDown(root)

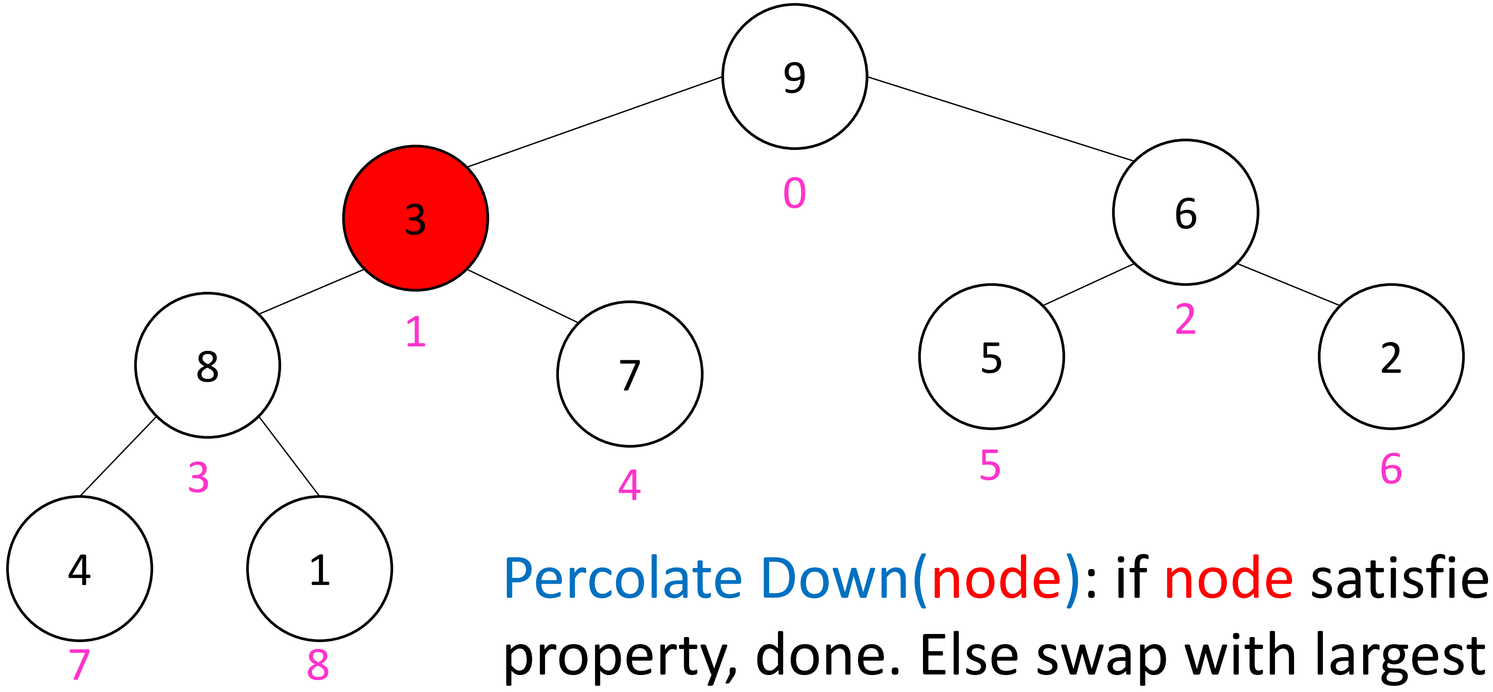


Percolate Down(node): if node satisfies heap property, done. Else swap with largest child and repeat on that subtree

Heap Sort

- Remove the Max element (i.e. the root) from the Heap: replace with last element, call percolateDown(root)

9	3	6	8	7	5	2	4	1	
0	1	2	3	4	5	6	7	8	9



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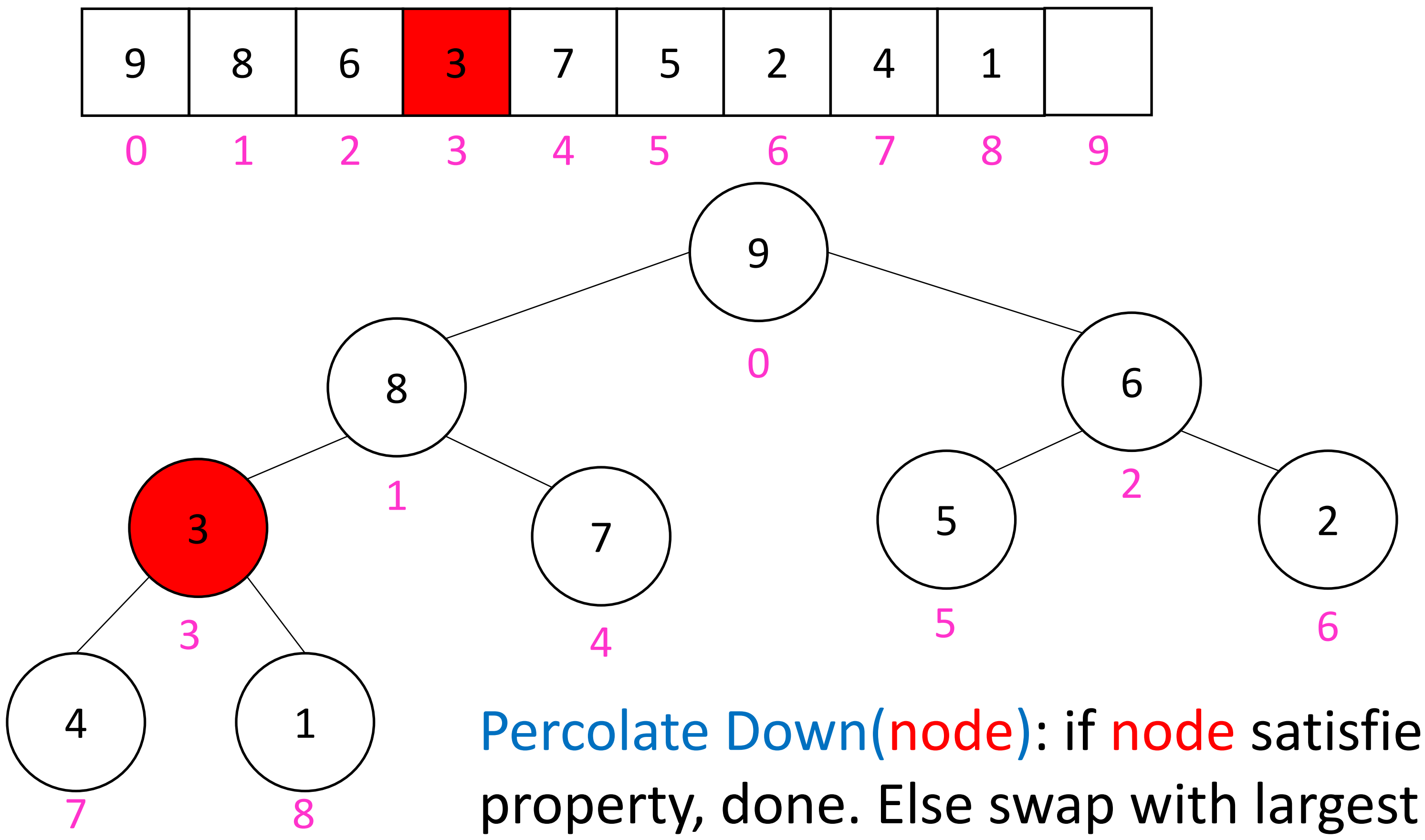

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Heap Sort

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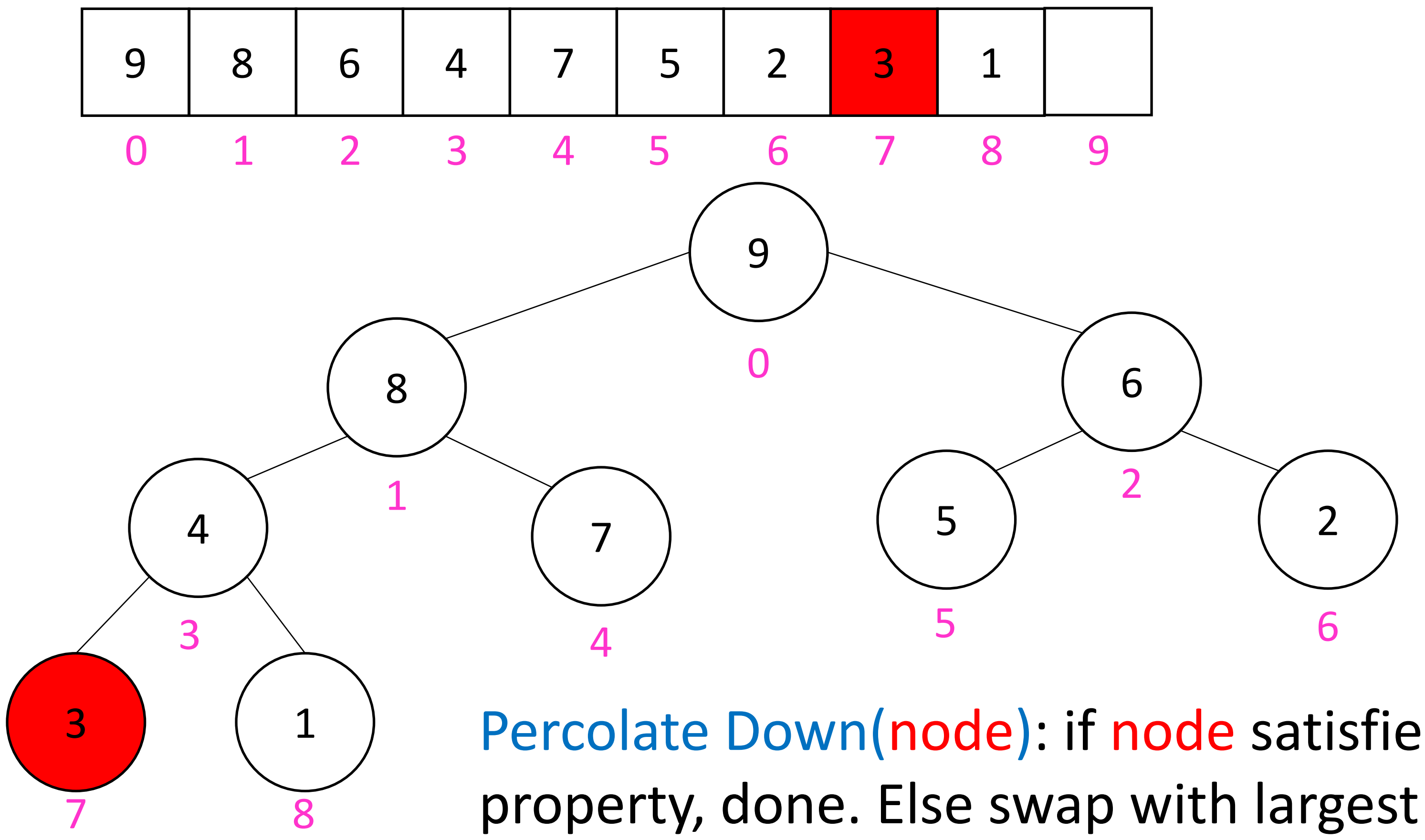


Percolate Down(node): if node satisfies heap property, done. Else swap with largest child and repeat on that subtree

Heap Sort

- Build a heap
- Call deleteMax
- Put that at the end of the array

```
myHeap = buildHeap(a);
for (int i = a.length-1; i>=0; i--){
	item = myHeap.deleteMax();
	a[i] = item;
}
```

Running Time:

Worst Case: $\Theta(\cdot)$

Best Case: $\Theta(\cdot)$

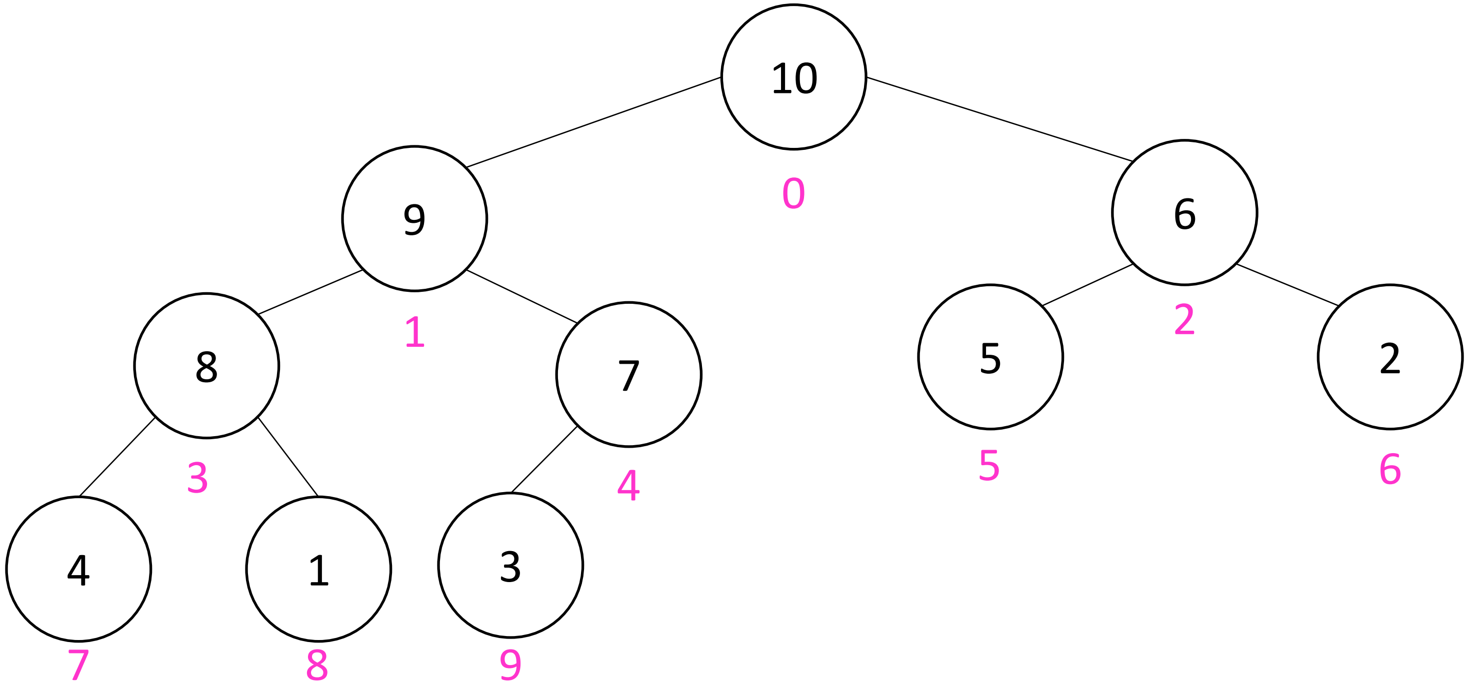
Properties of Heap Sort

- Worst Case Running time:
  - $\Theta(n \log n)$
- In-Place?
  - Not yet!
  - But in general, yes!
- Adaptive?
  - No
- Stable?
  - No

In Place Heap Sort

- Idea: When “removing” an element from the heap, swap it with the last item of the heap then “pretend” the heap is one item shorter

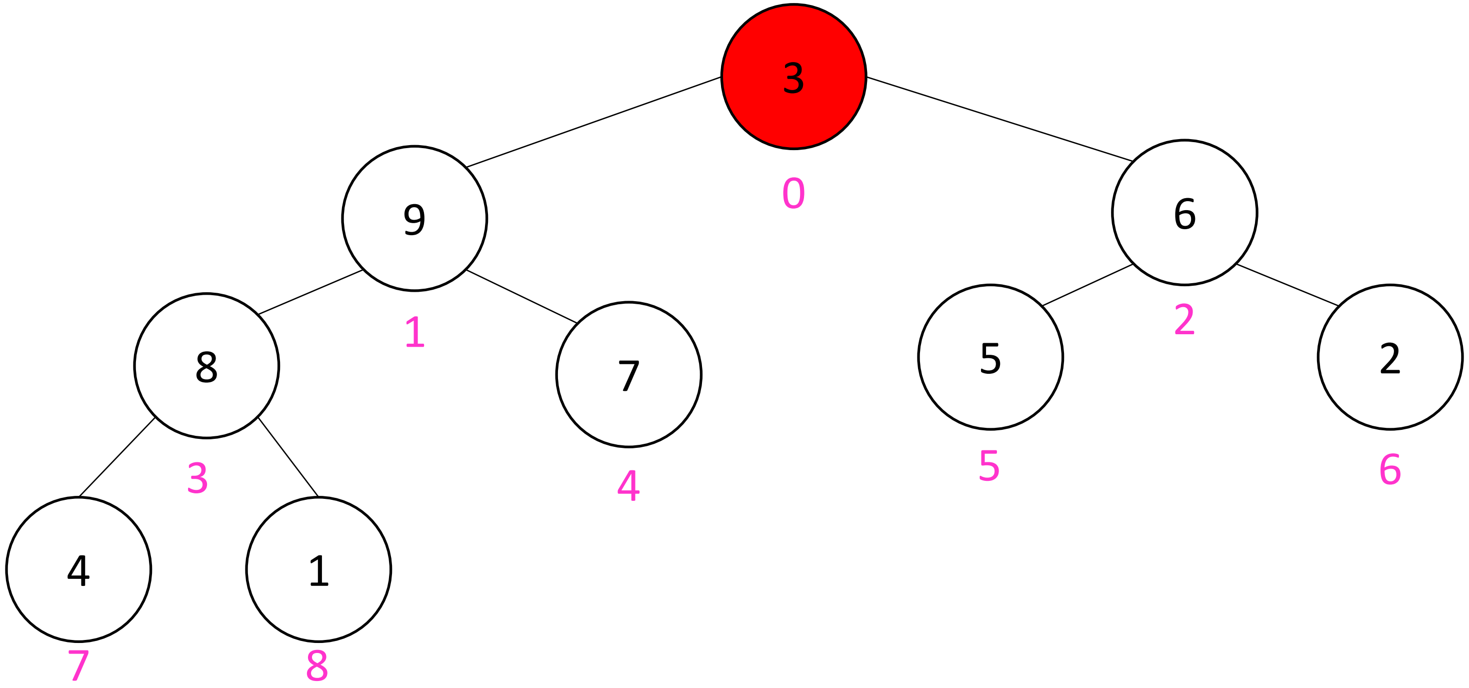
10	9	6	8	7	5	2	4	1	3
0	1	2	3	4	5	6	7	8	9



Heap Sort

- Idea: When “removing” an element from the heap, swap it with the last item of the heap then “pretend” the heap is one item shorter

3	9	6	8	7	5	2	4	1	10
0	1	2	3	4	5	6	7	8	9



Heap Sort

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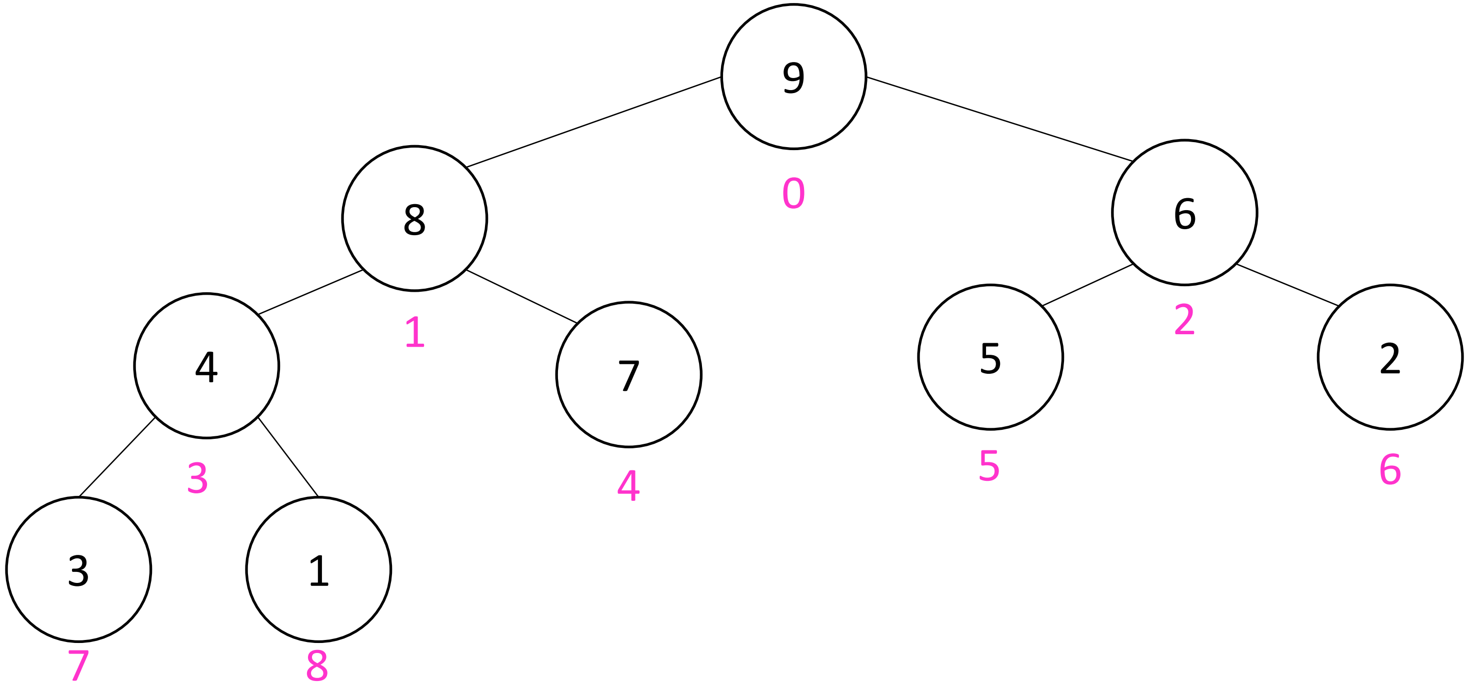
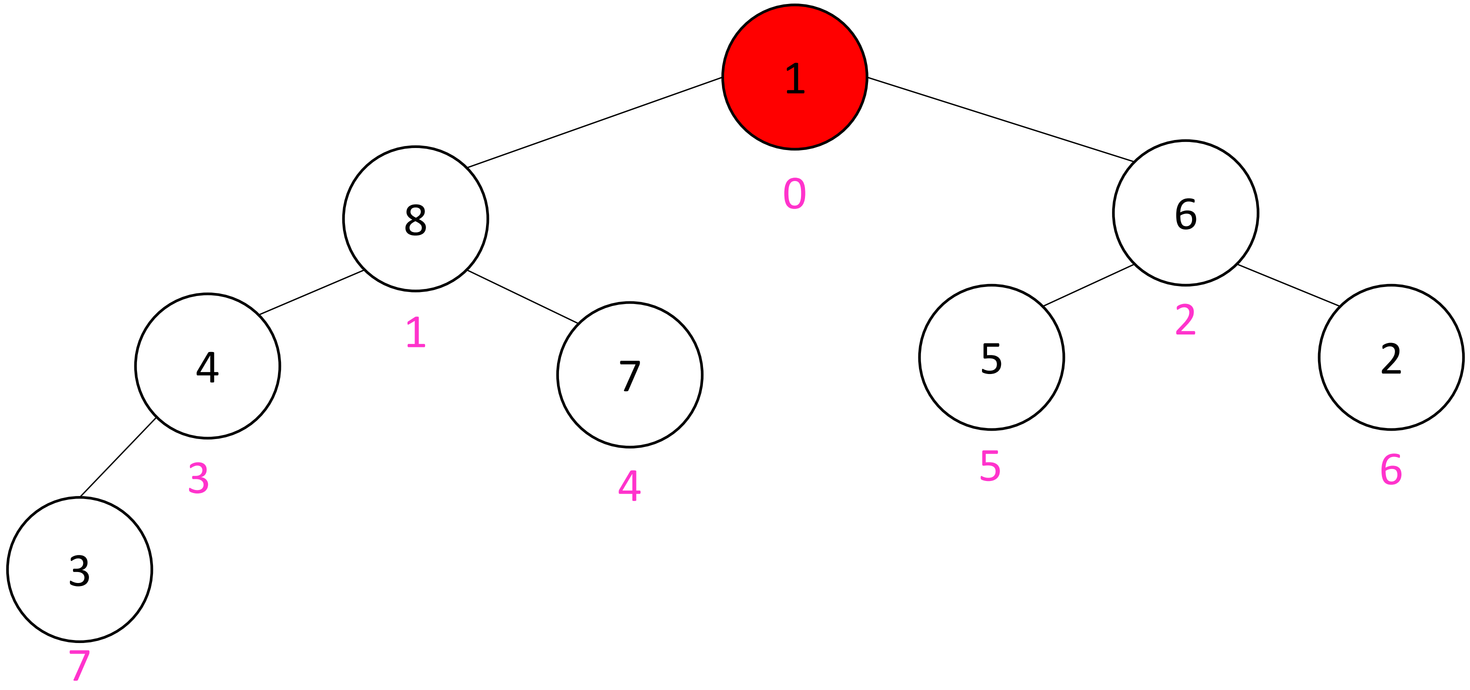


Heap Sort

- Idea: When "removing" an element from the heap, swap it with the last item of the heap then "pretend" the heap is one item shorter

1	8	6	4	7	5	2	3	9	10
0	1	2	3	4	5	6	7	8	9



Heap Sort

- Idea: When “removing” an element from the heap, swap it with the last item of the heap then “pretend” the heap is one item shorter

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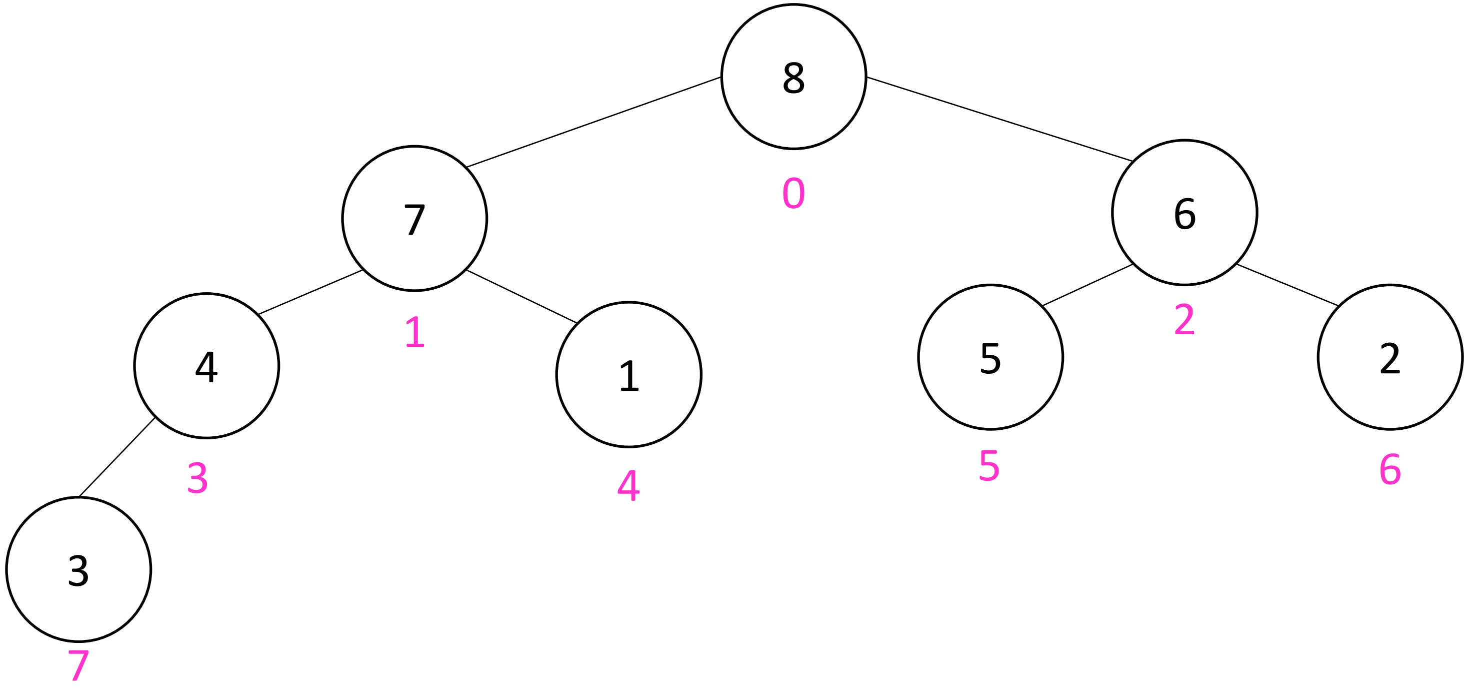
0	1	2	3	4	5	6	7	8	9



Heap Sort

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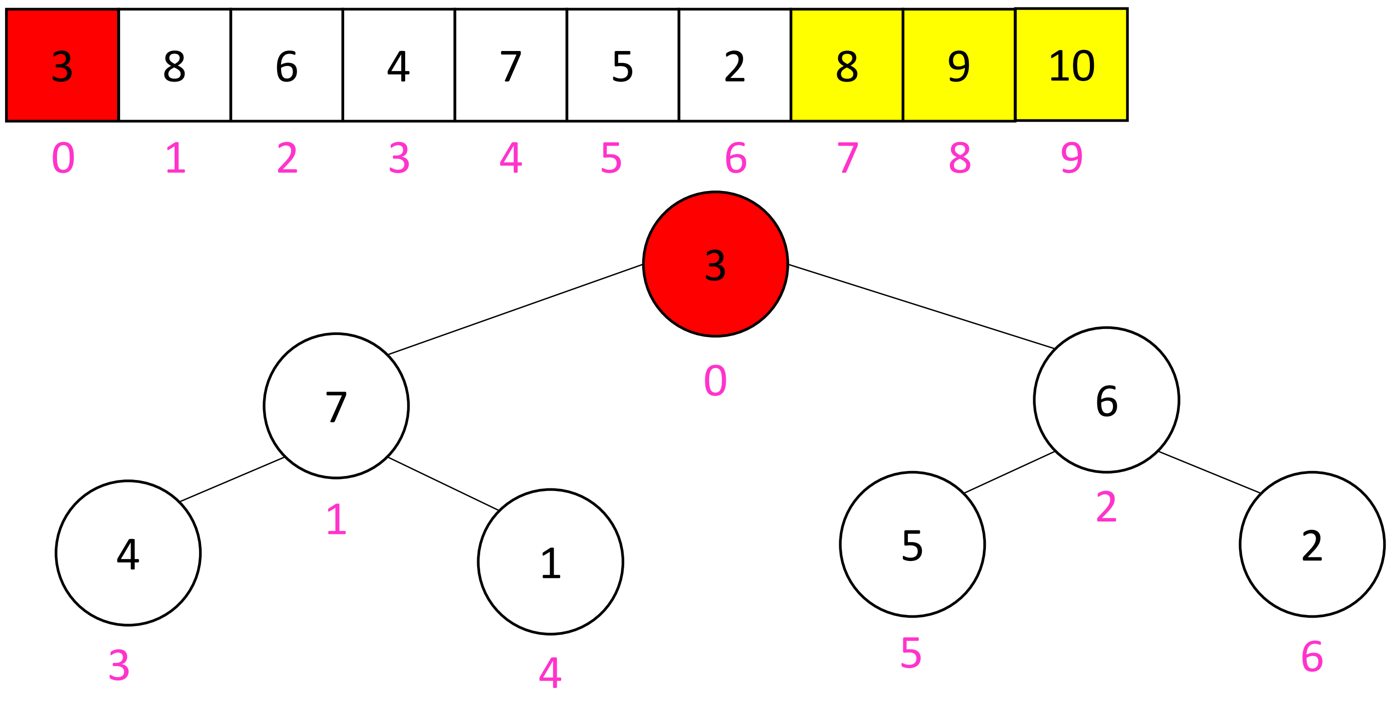
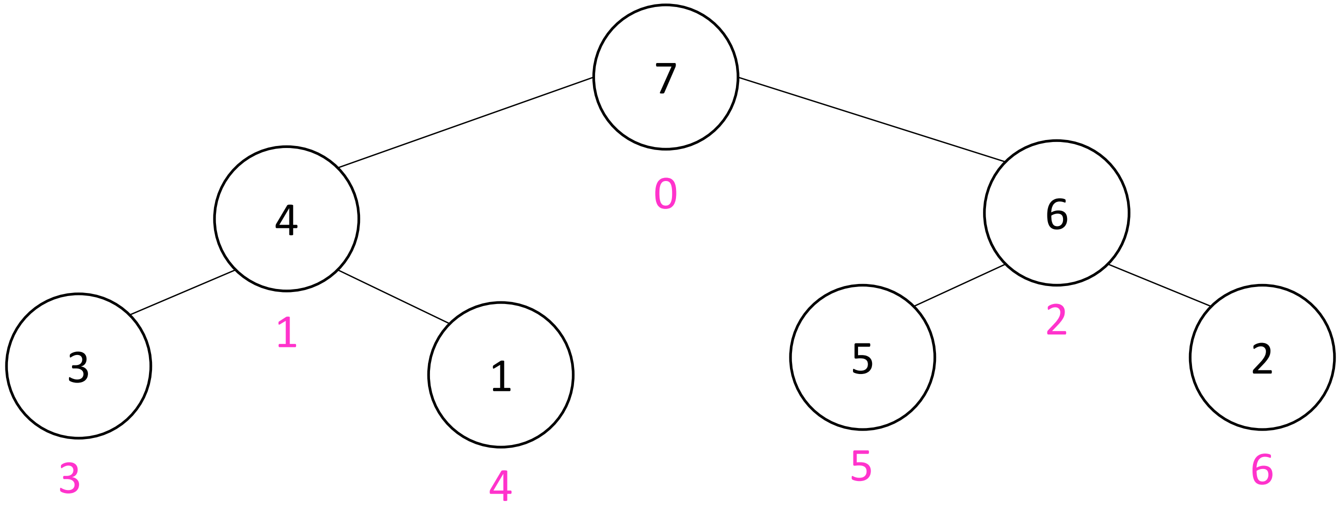


Heap Sort

- Idea: When "removing" an element from the heap, swap it with the last item of the heap then "pretend" the heap is one item shorter

3	8	6	4	7	5	2	8	9	10
0	1	2	3	4	5	6	7	8	9



In Place Heap Sort

- Build a heap using the same array (Floyd's build heap algorithm works)
- Call deleteMax
- Put that at the end of the array

```
buildHeap(a);
for (int i = a.length-1; i>=0; i--){
	temp=a[i]
	a[i] = a[0];
	a[0] = temp;
	percolateDown(0);
}
```

Running Time:

Worst Case: $\Theta(\cdot)$

Best Case: $\Theta(\cdot)$

Floyd’s buildHeap method

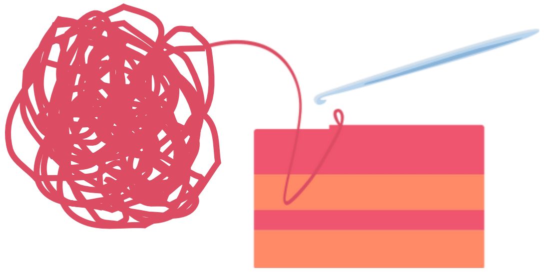
- Working towards the root, one row at a time, percolate down

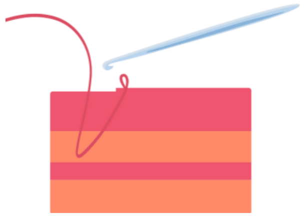
```
buildHeap(){
    for(int i = size; i>0; i--){
        percolateDown(i);
    }
}
```

Divide And Conquer Sorting

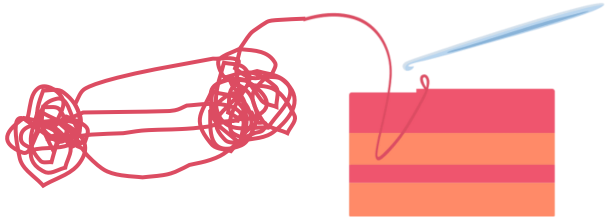
- Divide and Conquer:
  - Recursive algorithm design technique
  - Solve a large problem by breaking it up into smaller versions of the same problem

Divide and Conquer

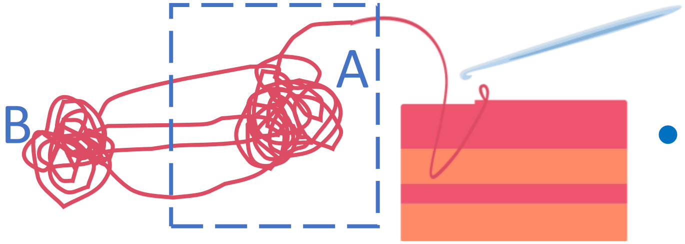




- **Base Case**:
  - If the problem is “small” then solve directly and return



- **Divide**:
  - Break the problem into subproblem(s), each smaller instances



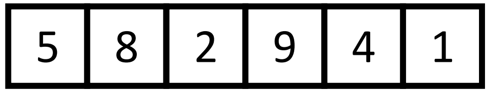
- **Conquer**:
  - Solve subproblem(s) recursively
- **Combine**:
  - Use solutions to subproblems to solve original problem

Divide and Conquer Template Pseudocode

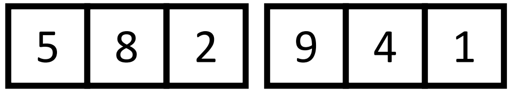
```
def my_DandC(problem){
    // Base Case
    if (problem.size() <= small_value){
        return solve(problem);  // directly solve (e.g., brute force)
    }
    // Divide
    List subproblems = divide(problem);

    // Conquer
    solutions = new List();
    for (sub : subproblems){
        subsolution = my_DandC(sub);
        solutions.add(subsolution);
    }
    // Combine
    return combine(solutions);
}
```

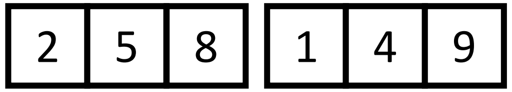
Merge Sort



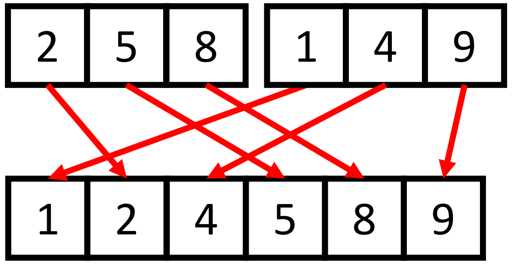
- **Base Case**:
  - If the list is of length 1 or 0, it’s already sorted, so just return it



- **Divide**:
  - Split the list into two “sublists” of (roughly) equal length



- **Conquer**:
  - Sort both lists recursively



- **Combine**:
  - **Merge** sorted sublists into one sorted list

Merge Sort In Action!

Sort between indices low and $high$

5	8	2	9	4	1	3	7

low … $high$

Base Case: if low == $high$ then that range is already sorted!

Divide and Conquer: Otherwise call mergesort on ranges $\left(low, \frac{low+high}{2}\right)$ and $\left(\frac{low+high}{2}+1, high\right)$

5	8	2	9	4	1	3	7

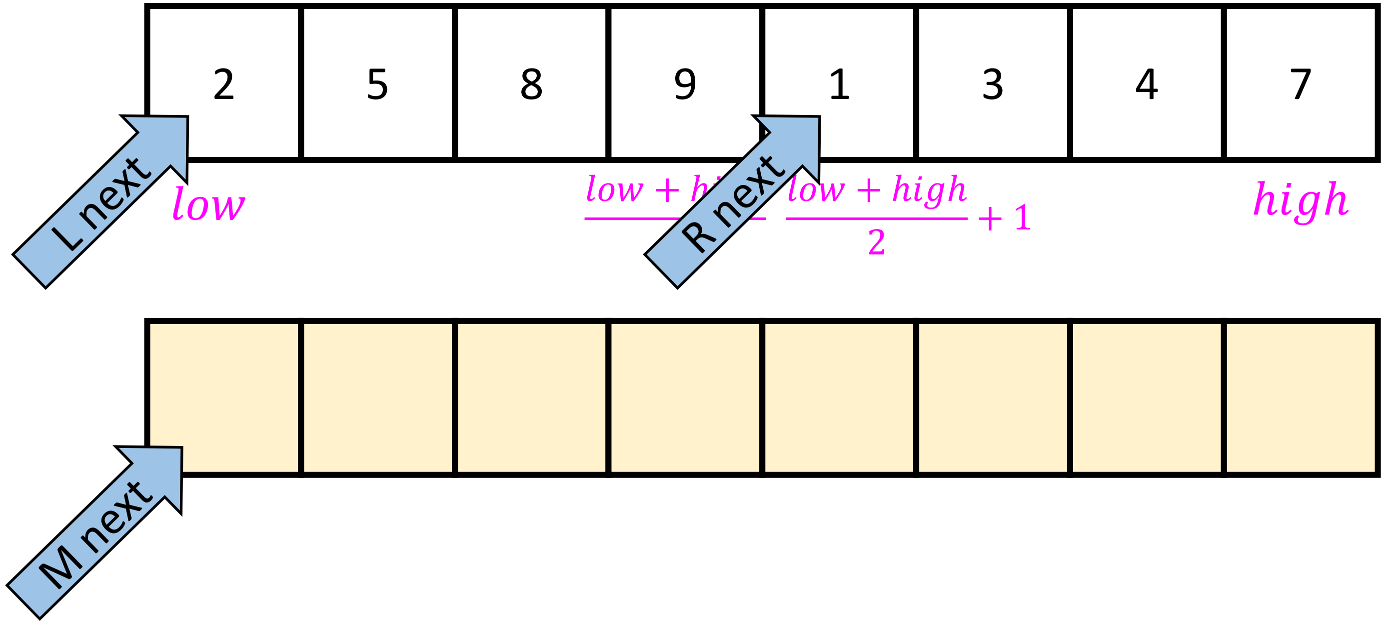
low … $\frac{low+high}{2}$ | $\frac{low+high}{2}+1$ … $high$

After Recursion:

2	5	8	9	1	3	4	7

low … $high$

Merge (the combine part)



Create a new array to merge into, and 3 pointers/indices:

- L_next: the smallest "unmerged" thing on the left
- R_next: the smallest "unmerged" thing on the right
- M_next: where the next smallest thing goes in the merged array

One-by-one: put the smallest of L_next and R_next into M_next, then advance both M_next and whichever of L/R was used.

Merge Sort Pseudocode

```
void mergesort(myArray){
        ms_helper(myArray, 0, myArray.length());
}
void mshelper(myArray, low, high){
        if (low == high){return;}  // Base Case
        mid = (low+high)/2;
        ms_helper(low, mid);
        ms_helper(mid+1, high);
        merge(myArray, low, mid, high);
}
```

Merge Pseudocode

```
void merge(myArray, low, mid, high){
        merged = new int[high-low+1]; // or whatever type is in myArray
        l_next = low;
        r_next = high;
        m_next = 0;
        while (l_next <= mid && r_next <= high){
                if (myArray[l_next] <= myArray[r_next]){
                        merged[m_next++] = myArray[l_next++];
                }
                else{
                        merged[m_next++] = myArray[r_next++];
                }
        }
        while (l_next <= mid){ merged[m_next++] = myArray[l_next++]; }
        while (r_next <= high){  merged[m_next++] = myArray[r_next++]; }
        for(i=0; i<=merged.length; i++){ myArray[i+low] = merged[i];}
}
```

Analyzing Merge Sort

1. Identify time required to Divide and Combine
2. Identify all subproblems and their sizes
3. Use recurrence relation to express recursive running time
4. Solve and express running time asymptotically

- **Divide**: 0 comparisons
- **Conquer**: recursively sort two lists of size $\frac{n}{2}$
- **Combine**: n comparisons
- **Recurrence:**

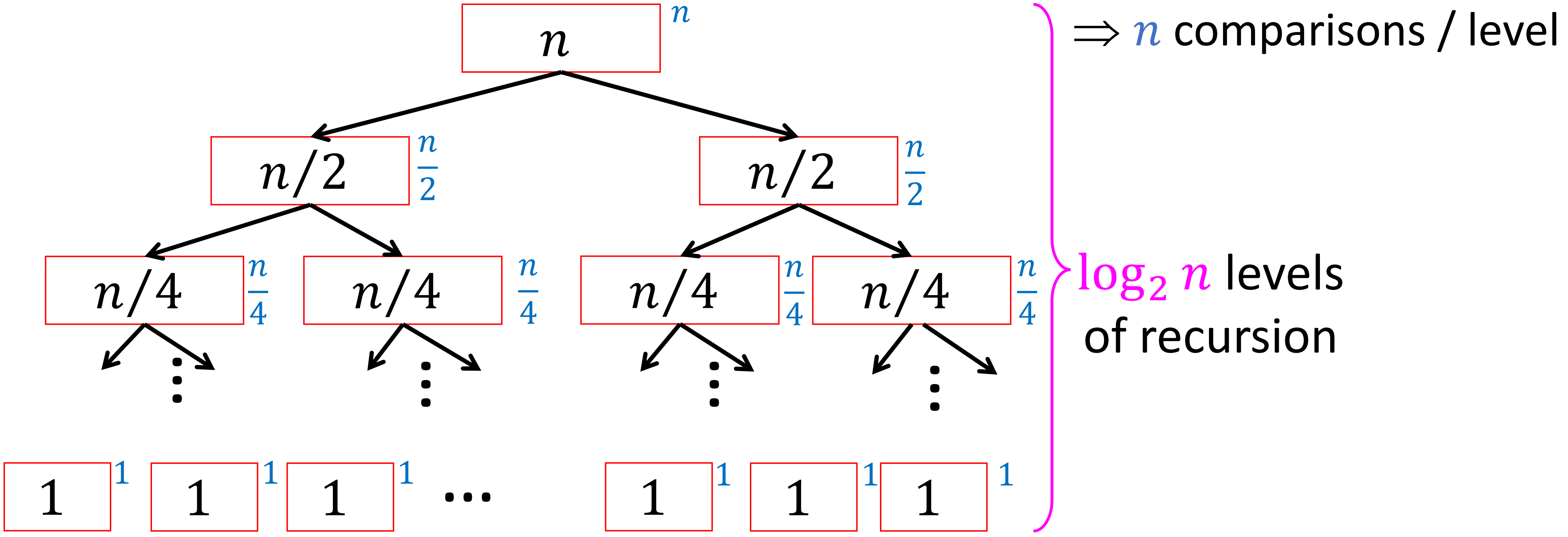
$$T(n) = 0 + T\left(\frac{n}{2}\right) + T\left(\frac{n}{2}\right) + n$$

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Red box represents a problem instance

Blue value represents time spent at that level of recursion

$$T(n) = 2T(\frac{n}{2}) + n$$



$$T(n) = \sum_{i=1}^{\log_2 n} n = n \log_2 n$$

Properties of Merge Sort

- Worst Case Running time:
  - $\Theta(n \log n)$
- In-Place?
  - No!
- Adaptive?
  - No!
- Stable?
  - Yes!
  - As long as in a tie you always pick l_next

Quicksort

- Like Mergesort:
    - Divide and conquer
    - $O(n \log n)$ run time (kind of…)
- Unlike Mergesort:
    - Divide step is the “hard” part
    - *Typically* faster than Mergesort

Quicksort

Idea: pick a pivot element, recursively sort two sublists around that element

- Divide: select pivot element p, Partition(p)
- Conquer: recursively sort left and right sublists
- Combine: Nothing!

Partition (Divide step)

Given: a list, a pivot p

Start: unordered list

8	5	7	3	12	10	1	2	4	9	6	11

Goal: All elements $< p$ on left, all $> p$ on right

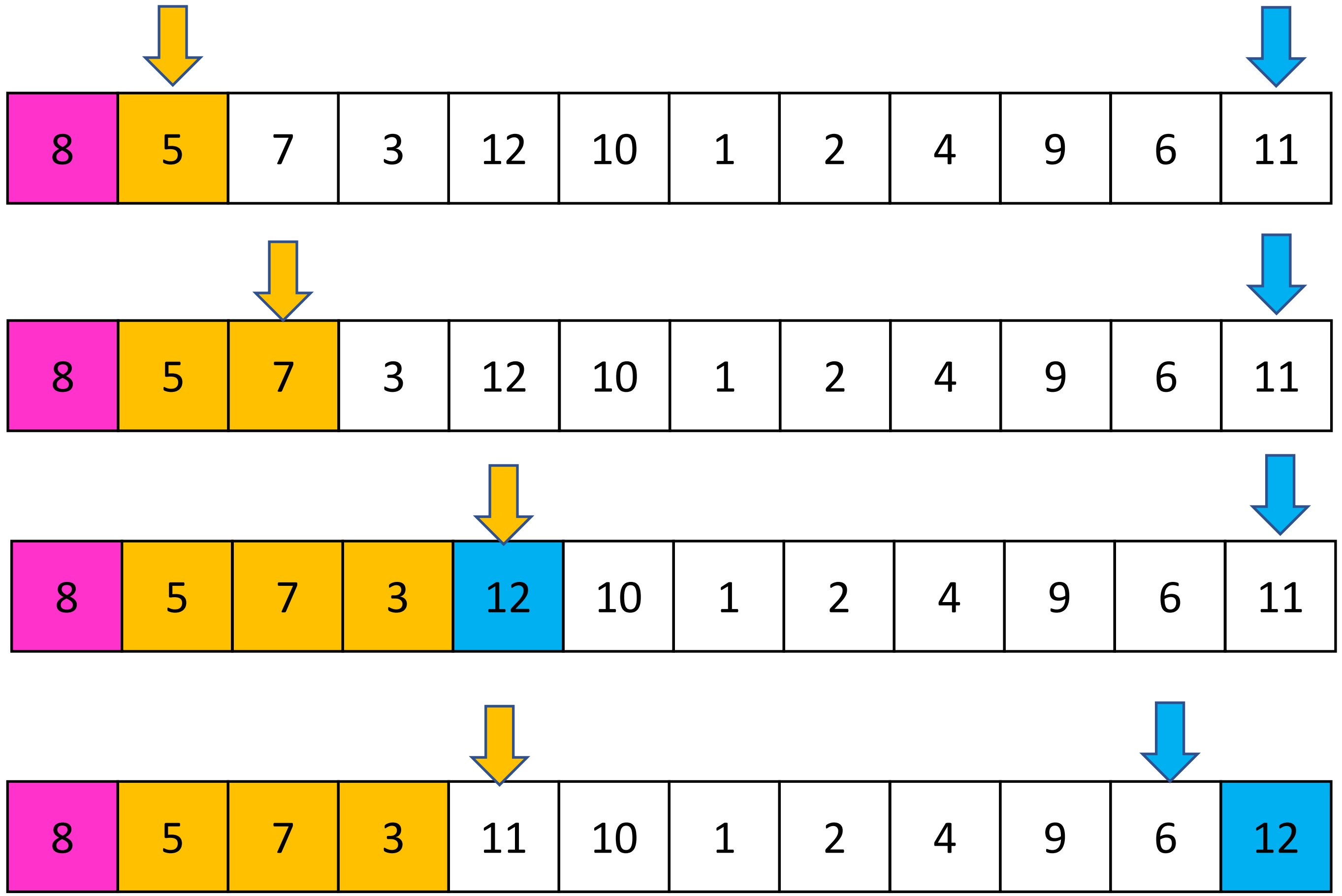
5	7	3	1	2	4	6	8	12	10	9	11

Partition, Procedure

If Begin value $<\ p$, move Begin right

Else swap Begin value with End value, move End Left

Done when Begin = End



Partition, Procedure

If Begin value $<$ p, move Begin right

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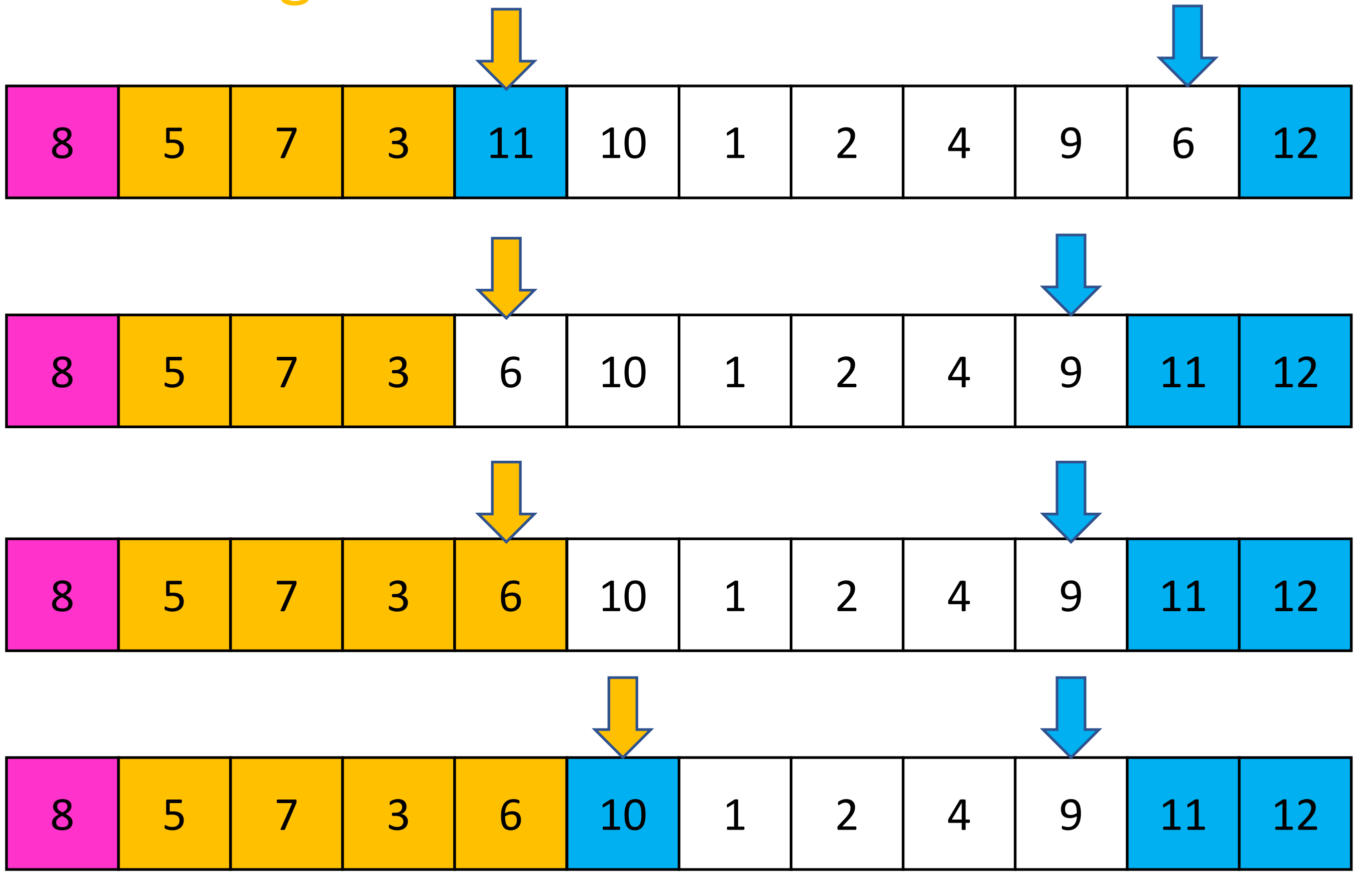
Done when Begin = End

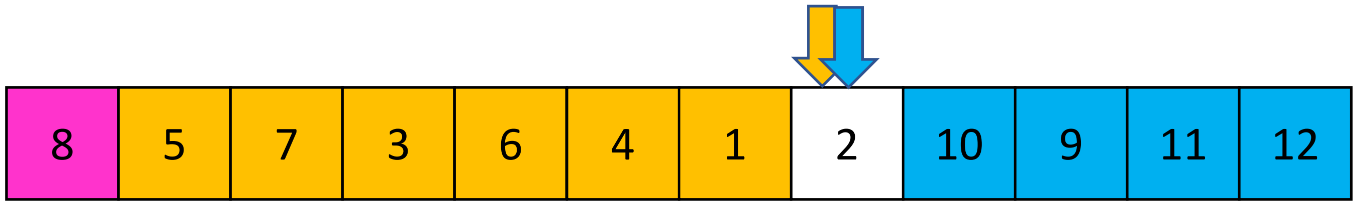


Partition, Procedure

If Begin value $<$ p, move Begin right

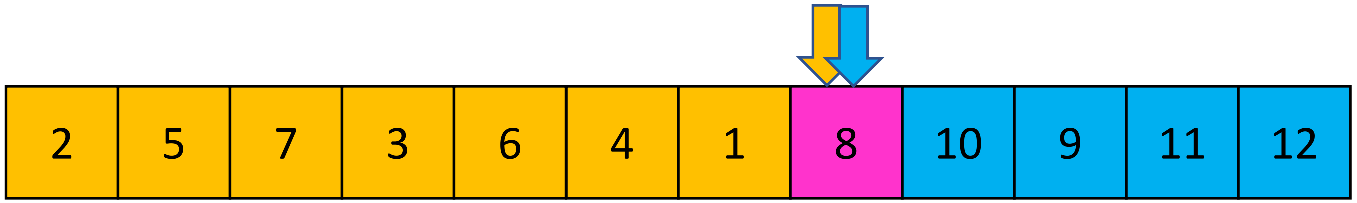
Else swap Begin value with End value, move End Left

Done when Begin = End



Case 1: meet at element $< p$

Swap p with pointer position (2 in this case)

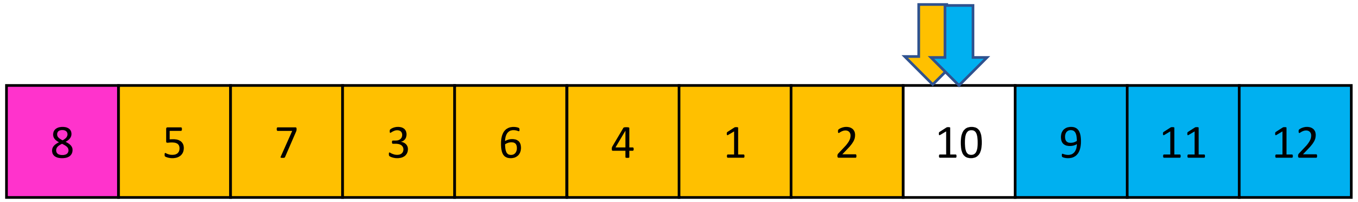


Partition, Procedure

If Begin value $<\ p$, move Begin right

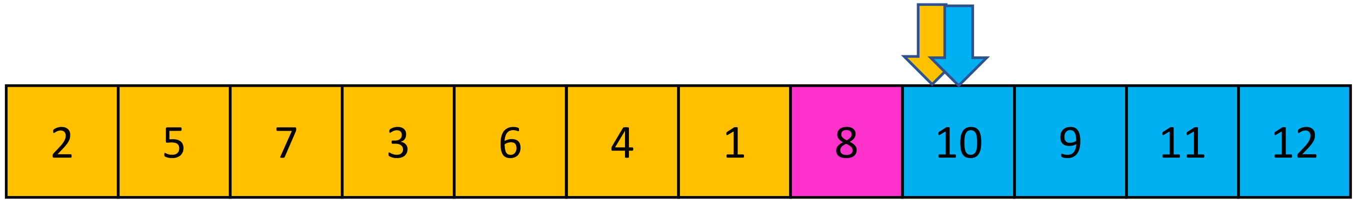
Else swap Begin value with End value, move End Left

Done when Begin = End



Case 2: meet at element $> p$

Swap p with value to the left (2 in this case)

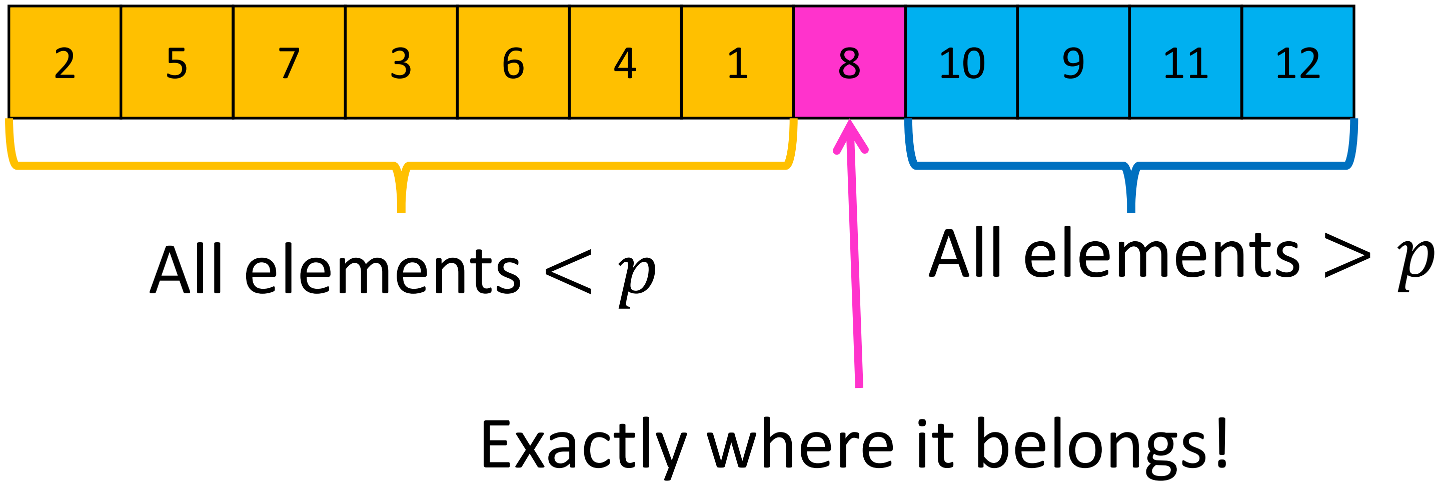


Partition Summary

1. Put p at beginning of list
2. Put a pointer (Begin) just after p, and a pointer (End) at the end of the list
3. While Begin < End:
    1. If Begin value $<\ p$, move Begin right
    2. Else swap Begin value with End value, move End Left
4. If pointers meet at element $< p$: Swap p with pointer position
5. Else If pointers meet at element $> p$: Swap p with value to the left

Run time? $O(n)$

Conquer



Recursively sort Left and Right sublists

Quicksort Run Time (Best)

If the pivot is always the median:

2	5	1	3	6	4	7	8	10	9	11	12

2	1	3	5	6	4	7	8	9	10	11	12

Then we divide in half each time

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

$$T(n) = O(n \log n)$$

Quicksort Run Time (Worst)

If the pivot is always at the extreme:

1	5	2	3	6	4	7	8	10	9	11	12

1	2	3	5	6	4	7	8	10	9	11	12

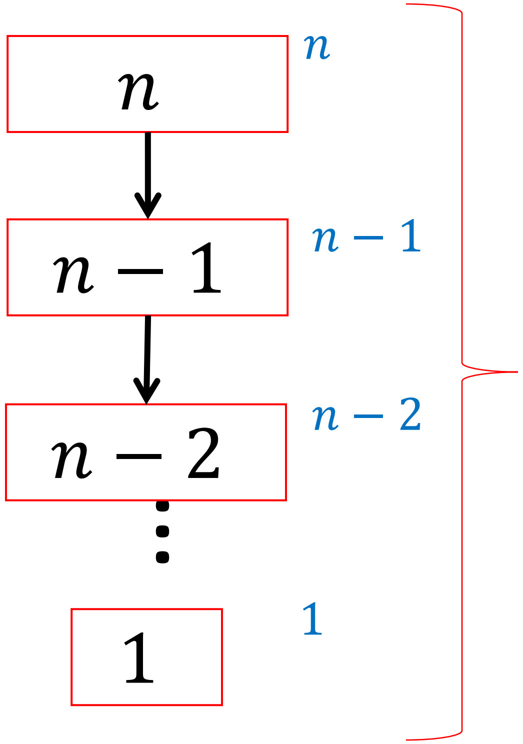
Then we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Quicksort Run Time (Worst)

$$T(n) = T(n-1) + n$$



$$T(n) = 1 + 2 + 3 + \cdots + n$$

$$T(n) = \frac{n(n+1)}{2}$$

$$T(n) = O(n^2)$$

Quicksort on a (nearly) Sorted List

First element always yields unbalanced pivot

1	2	3	4	5	6	7	8	9	10	11	12

1	2	3	4	5	6	7	8	9	10	11	12

So we shorten by 1 each time

$$T(n) = T(n-1) + n$$

$$T(n) = O(n^2)$$

Good Pivot

- What makes a good Pivot?
  - Roughly even split between left and right
  - Ideally: median
- There are ways to find the median in linear time, but it’s complicated and slow and you’re better off using mergesort
- In Practice:
  - Pick a random value as a pivot
  - Pick the middle of 3 random values as the pivot

Properties of Quick Sort

- Worst Case Running time:
  - $\Theta(n^2)$
  - But $\Theta(n \log n)$ average! And typically faster than mergesort!
- In-Place?
  - ….Debatable
- Adaptive?
  - No!
- Stable?
  - No!

Improving Running time

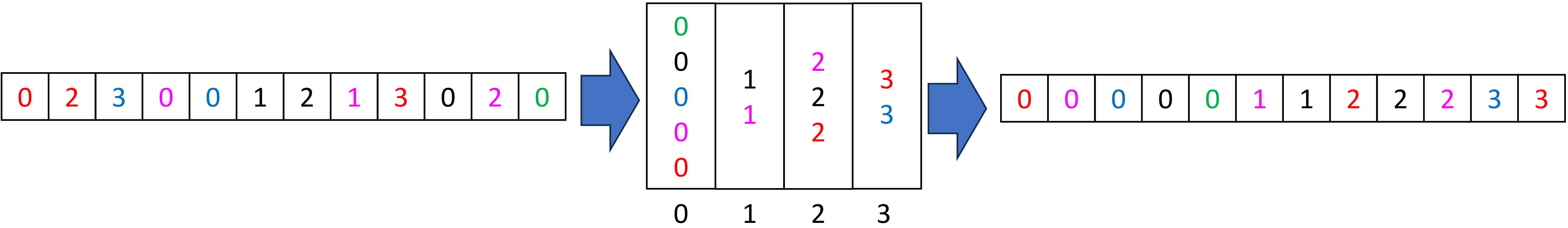
- Recall our definition of the sorting problem:
  - Input:
    - An array A of items
    - A comparison function for these items
      - Given two items x and y, we can determine whether $x < y$, $x > y$, or $x = y$
  - Output:
    - A permutation of A such that if $i \leq j$ then $A[i] \leq A[j]$
- Under this definition, it is impossible to write an algorithm faster than $n \log n$ asymptotically.
- Observation:
  - Sometimes there might be ways to determine the position of values without comparisons!

“Linear Time” Sorting Algorithms

- Useable when you are able to make additional assumptions about the contents of your list (beyond the ability to compare)
  - Examples:
    - The list contains only positive integers less than k
    - The number of distinct values in the list is much smaller than the length of the list
- The running time expression will always have a term other than the list’s length to account for this assumption
  - Examples:
    - Running time might be $\Theta(k \cdot n)$ where k is the range/count of values

BucketSort

- Assumes the array contains integers between 0 and $k-1$ (or some other small range)
- Idea:
  - Use each value as an index into an array of size k
  - Add the item into the “bucket” at that index (e.g. linked list)
  - Get sorted array by “appending” all the buckets



BucketSort Running Time

- Create array of k buckets
  - Either $\Theta(k)$ or $\Theta(1)$ depending on some things…
- Insert all n things into buckets
  - $\Theta(n)$
- Empty buckets into an array
  - $\Theta(n+k)$
- Overall:
  - $\Theta(n+k)$
- When is this better than mergesort?

Properties of BucketSort

- In-Place?
  - No
- Adaptive?
  - No
- Stable?
  - Yes!

RadixSort

- Radix: The base of a number system
  - We'll use base 10, most implementations will use larger bases
- Idea:
  - BucketSort by each digit, one at a time, from least significant to most significant

103	801	401	323	255	823	999	101	113	901	555	512	245	800	018	121
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Place each element into a "bucket" according to its 1's place

800	801 401 101 901 121	512	103 323 823 113		255 555 245			018	999
0	1	2	3	4	5	6	7	8	9

RadixSort

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0	1	2	3	4	5	6	7	8	9
800	801 401 101 901 121	512	103 323 823 113		255 555 245			018	999

Place each element into a "bucket" according to its 10's place

0	1	2	3	4	5	6	7	8	9
800 801 401 101 901 103	512 113 018	121 323 823		245	255 555				999

RadixSort

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  - We'll use base 10, most implementations will use larger bases
- Idea:
  - BucketSort by each digit, one at a time, from least significant to most significant

0	1	2	3	4	5	6	7	8	9
800 801 401 101 901 103	512 113 018	121 323 823		245	255 555				999

Place each element into a "bucket" according to its 100's place

0	1	2	3	4	5	6	7	8	9
018	101 103 113 121	245 255	323	401	512 555			800 801 823	901 999

RadixSort

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  - We'll use base 10, most implementations will use larger bases
- Idea:
  - BucketSort by each digit, one at a time, from least significant to most significant

018	101 103 113 121	245 255	323	401	512 555			800 801 823	901 999
0	1	2	3	4	5	6	7	8	9

Convert back into an array

018	811	103	113	121	245	255	323	401	512	555	800	801	823	901	999
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

RadixSort Running Time

- Suppose largest value is m
- Choose a radix (base of representation) b
- BucketSort all n things using b buckets
  - $\Theta(n + k)$
- Repeat once per each digit
  - $\log_b m$ iterations
- Overall:
  - $\Theta(n \log_b m + b \log_b m)$
- In practice, you can select the value of b to optimize running time
- When is this better than mergesort?