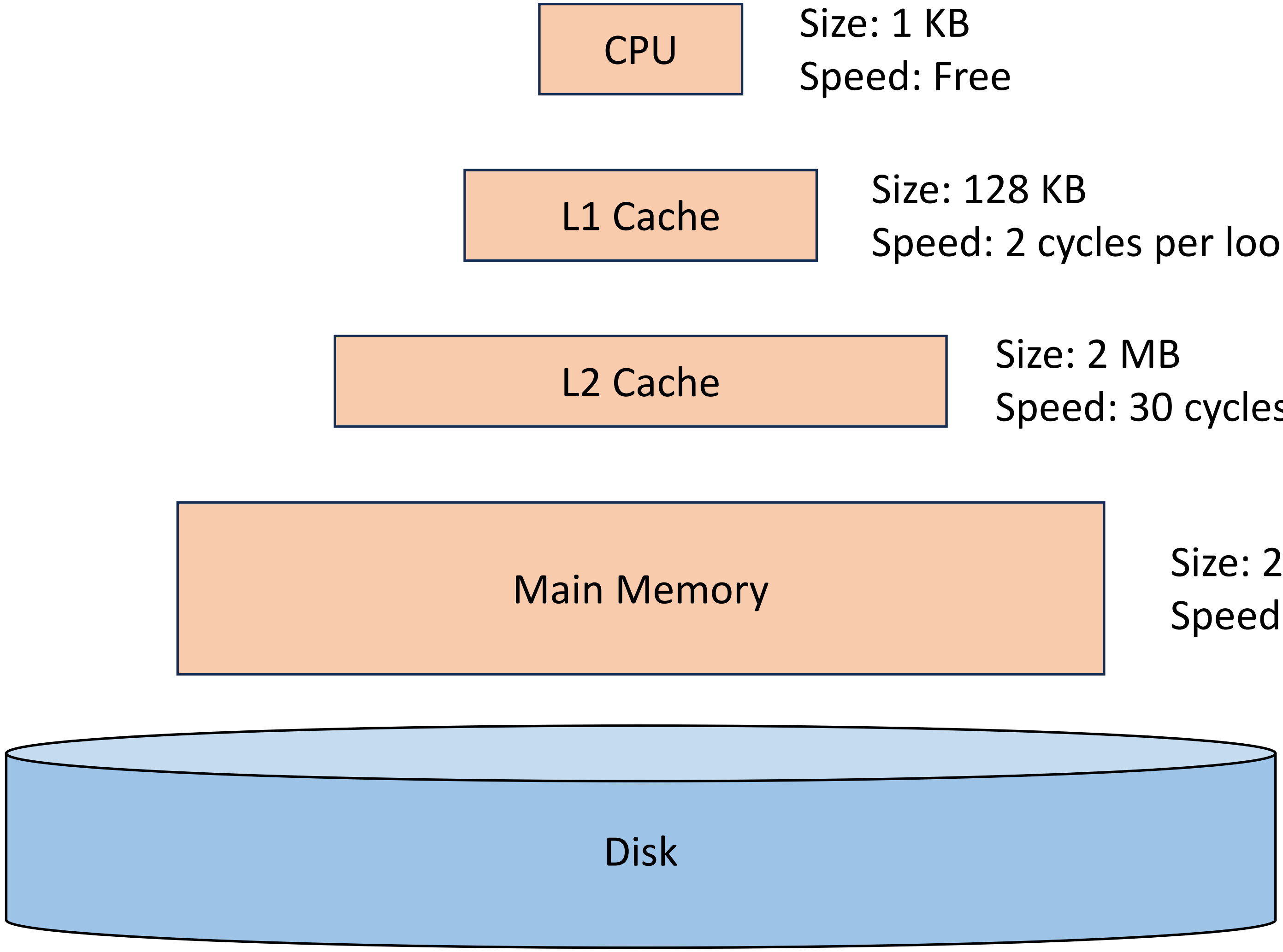
CSE 332 Winter 2024
Lecture 10: B Trees and Hashing

Nathan Brunelle

http://www.cs.uw.edu/332

Memory Hierarchy

Large memory is slow to access.
We use different “layers” or memory to balance quantity of storage with speed of access.
Ideally the data we want is in small memory, but we may need to go deep



CPU — Size: 1 KB
Speed: Free

L1 Cache — Size: 128 KB
Speed: 2 cycles per lookup

L2 Cache — Size: 2 MB
Speed: 30 cycles per lookup

Main Memory — Size: 2 GB
Speed: 250 cycles per lookup

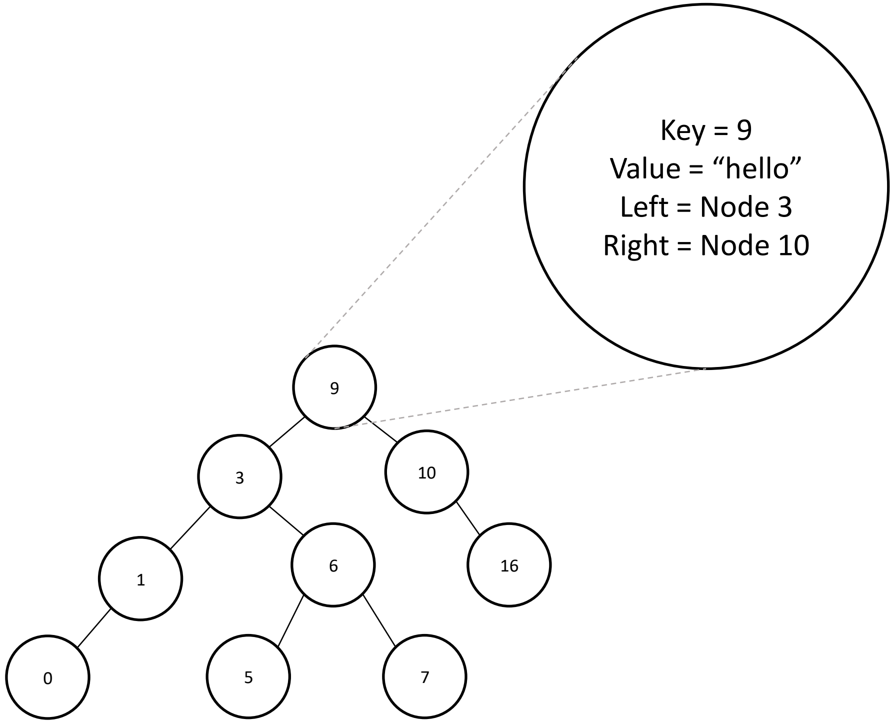
Disk — Size: 1 TB
Speed: 8,000,000 cycles per lookup

B Trees Motivation

- Memory Locality
  - Observation: in practice, when you read from memory you're likely to soon thereafter read from nearby memory
  - When memory is "fetched", it's collected in blocks at a time
  - Works well for arrays (they're contiguous is memory)
  - May not be helpful for linked lists, BSTs, etc. (pointers could go wherever)
- Solution: Have a BST-like data structure which can take advantage of locality

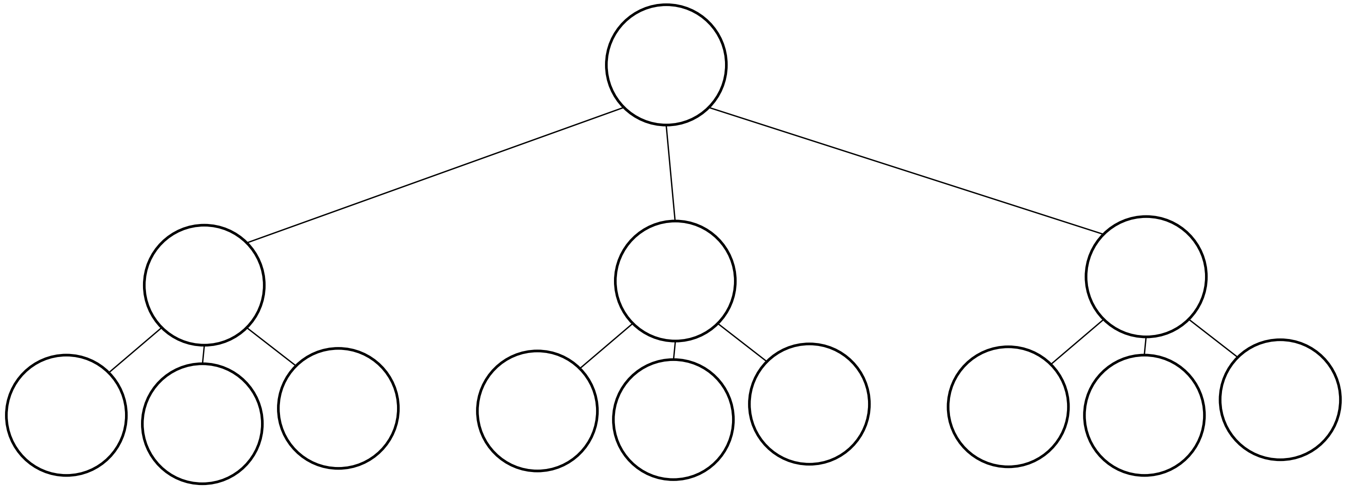
First Idea

- BST nodes have a lot of information inside them
- We don’t need that information for “intermediate” nodes
- Solution: Delay loading anything except keys as long as possible

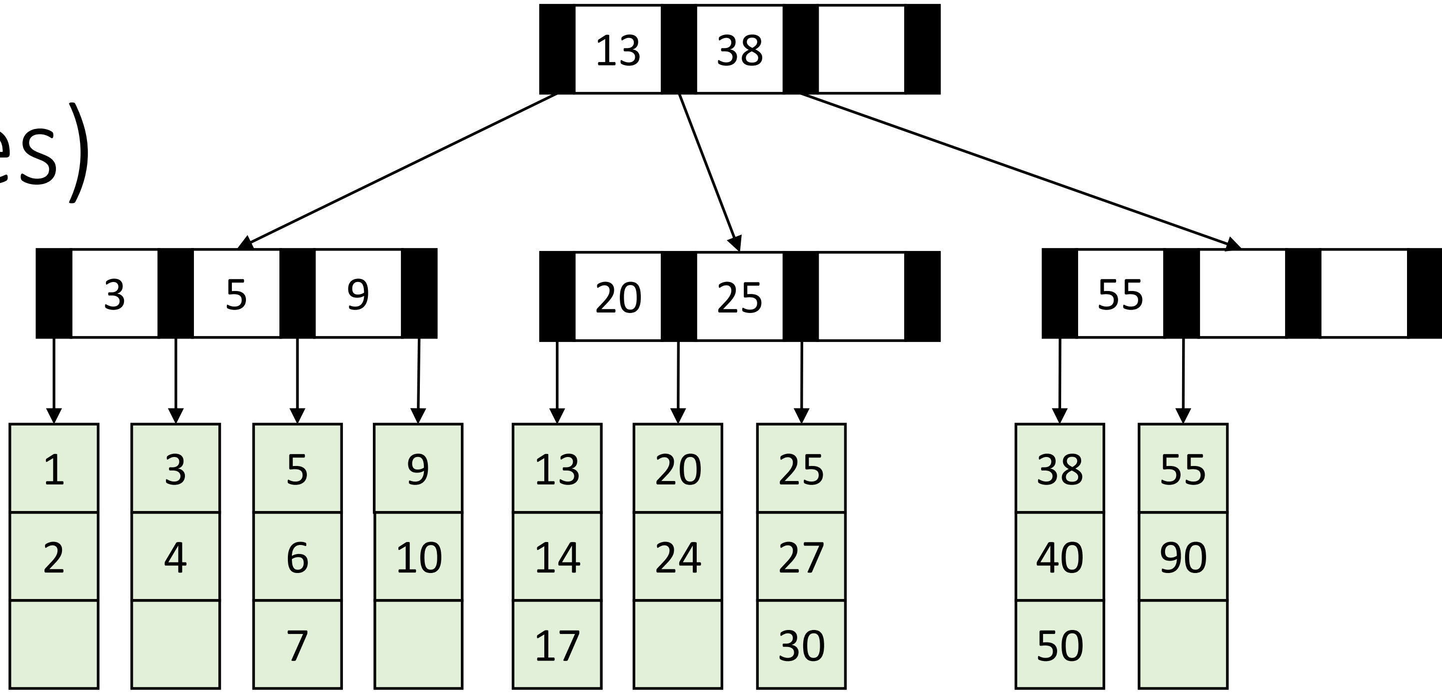


Second Idea

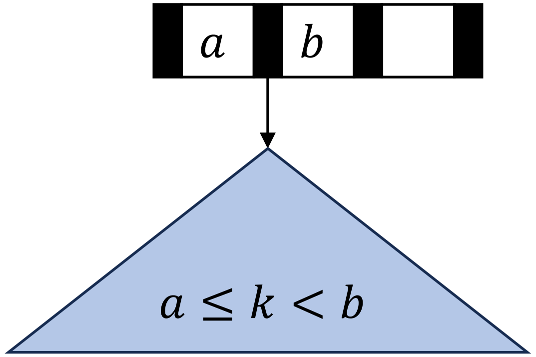
- Nodes may not be close to each other in memory
- In the worst case, each step in a traversal could go deep in memory
- Solution: Increase branching factor of tree, load blocks of keys at a time
  - M-ary tree: each node has at most M children
  - Choose M to snugly fit in a block



B Trees (aka B+ Trees)

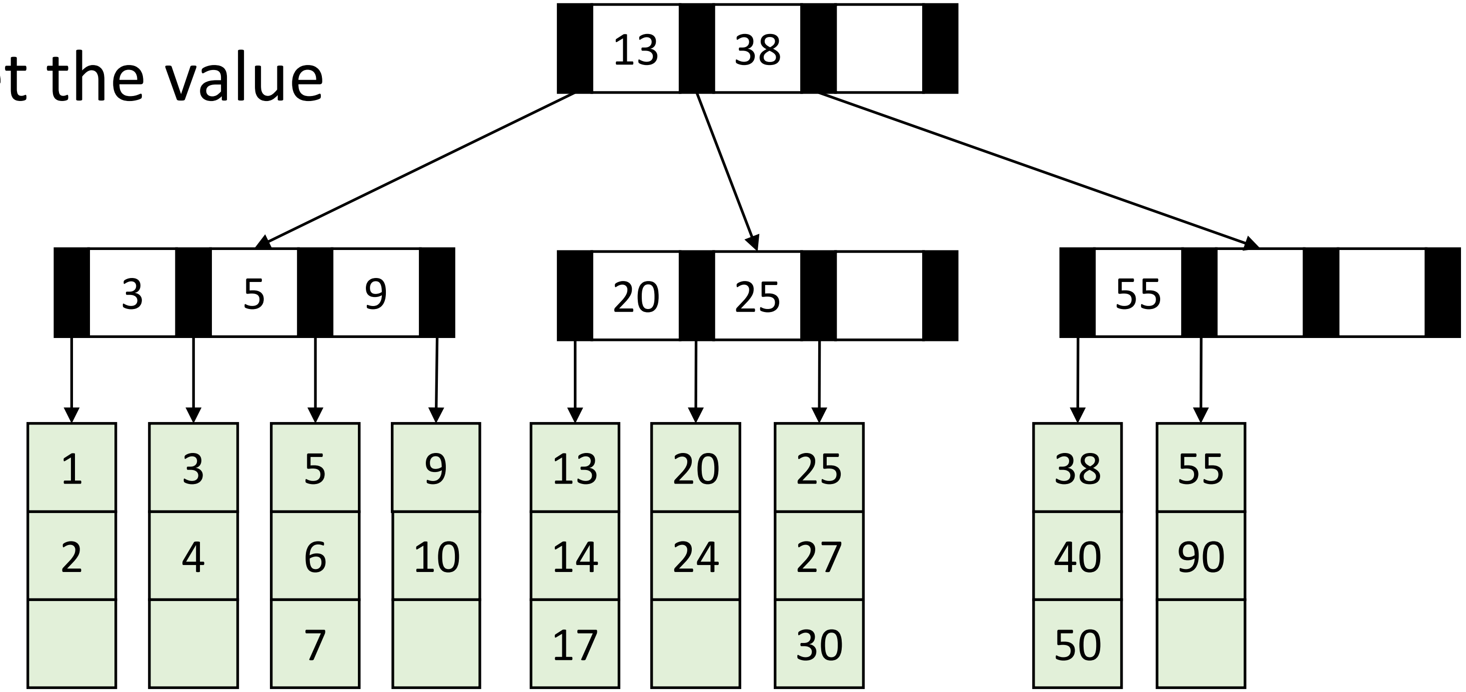


- Two types of nodes:
  - Internal Nodes
    - Sorted array of $M-1$ keys
    - Has M children
    - No other data!
  - Leaf Nodes
    - Sorted array of L key-value pairs
- Subtree between values a and b must contain only keys that are $\geq a$ and $< b$
  - If a is missing use $-\infty$
  - If b is missing use ∞



Find

- Start at the root node
- Binary search internal nodes to identify correct subtree
- Repeat until you reach a leaf node
- Binary search the leaf to get the value



Aside: Implementation

- What an internal node class might look like:
  - int M
  - int[] keys
  - Node[] children
  - int num_children
- What a leaf node class might look like:
  - int L
  - E[] data
  - int num_items

B Tree Structure Requirements

- Root:
  - If the tree has $\leq L$ items then root is a leaf node
  - Otherwise it is an internal node
- Internal Nodes:
  - Must have at least $\left\lceil \frac{M}{2} \right\rceil$ children (at least half full)
    - Unless it's the root and there aren't enough items to have that many children
- Leaf Nodes:
  - Must have at least Must have at least $\left\lceil \frac{L}{2} \right\rceil$ items (at least half full)
    - Unless it's the root and there aren't at least $\left\lceil \frac{L}{2} \right\rceil$ items
  - All leaves are at the same depth

Insertion Summary

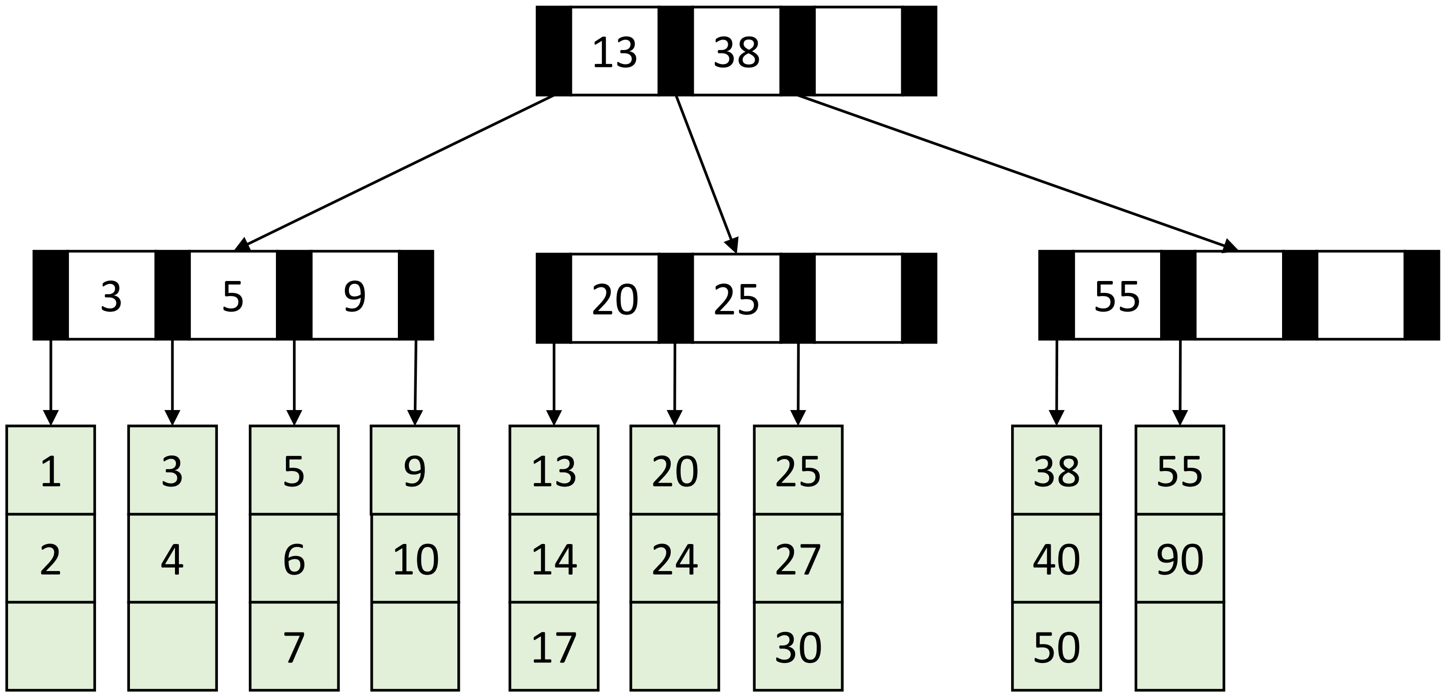
- Binary search to find which leaf should contain the new item
- If there's room, add it to the leaf array (maintaining sorted order)
- If there's not room, **split**
    - Make a new leaf node, move the larger $\left\lfloor \frac{L+1}{2} \right\rfloor$ items to it
    - If there's room in the parent internal node, add new leaf to it (with new key bound value)
    - If there's not room in the parent internal node, **split** that!
        - Make a new internal node and have it point to the larger $\left\lfloor \frac{M+1}{2} \right\rfloor$
        - If there's room in the parent internal node, add this internal node to it
        - If there's not room, repeat this process until there is!

Insertion TLDR

- Find where the item goes by repeated binary search
- If there’s room, just add it
- If there’s not room, split things until there is

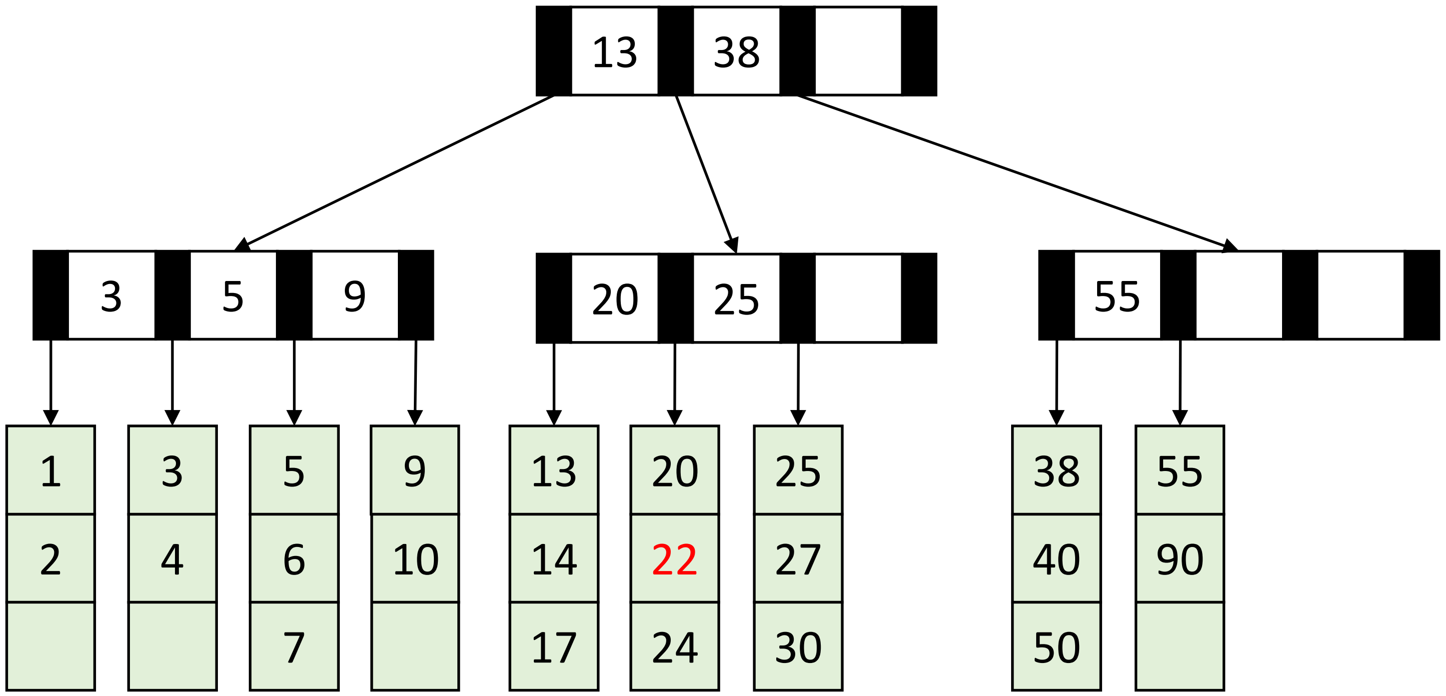
Insert Example

Insert 22



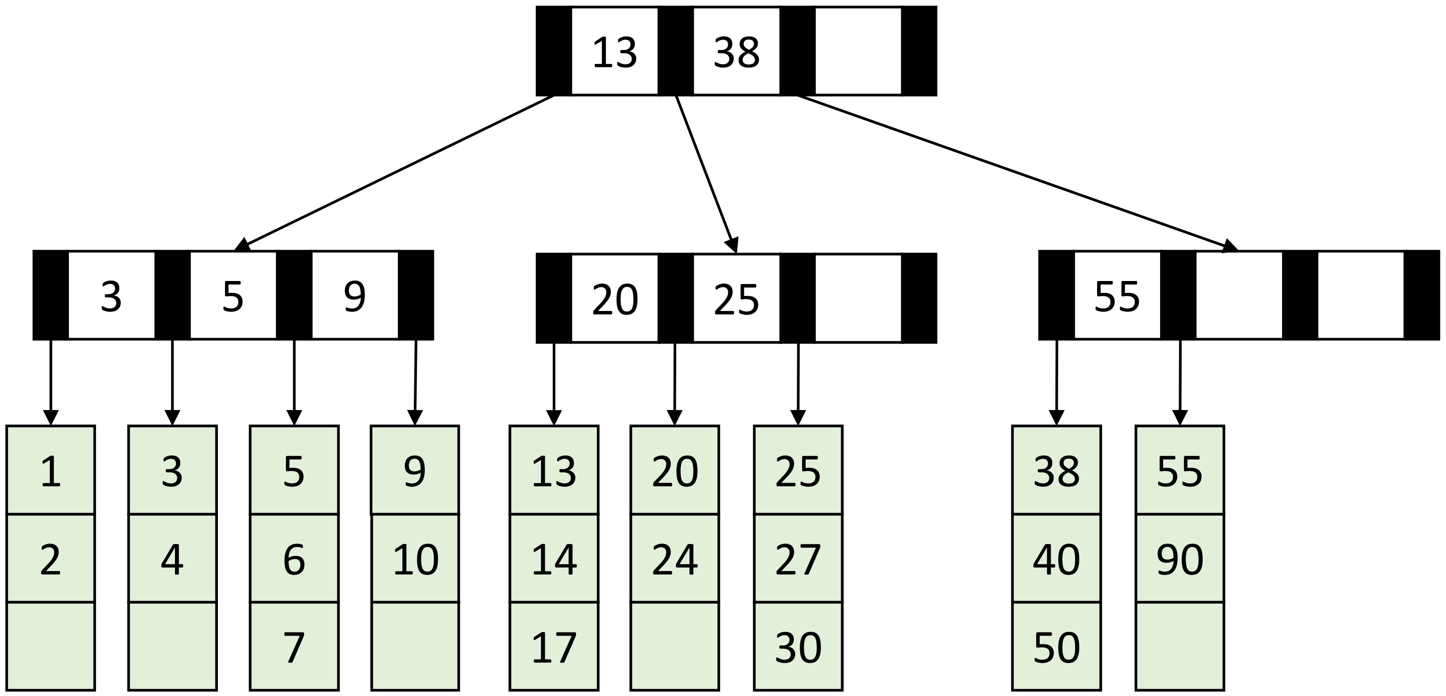
Insert Example

Insert 22



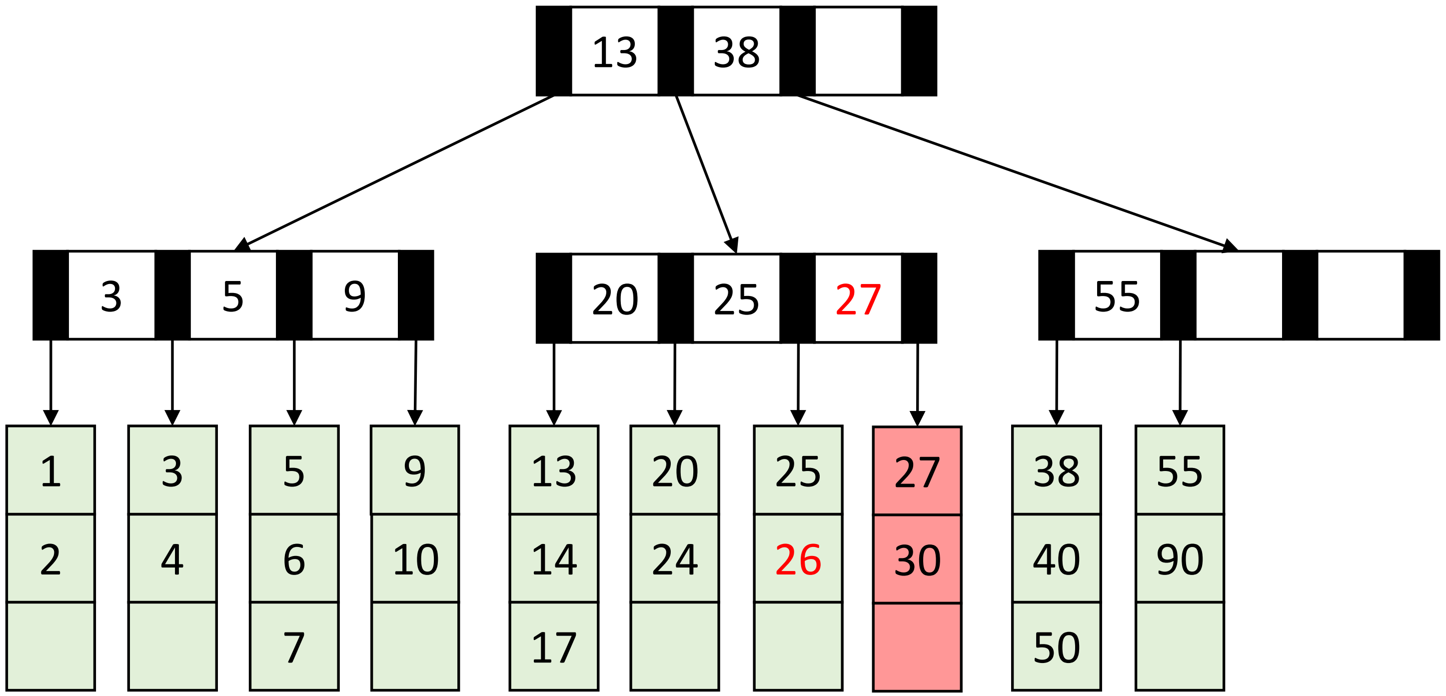
Insert Example

Insert 26



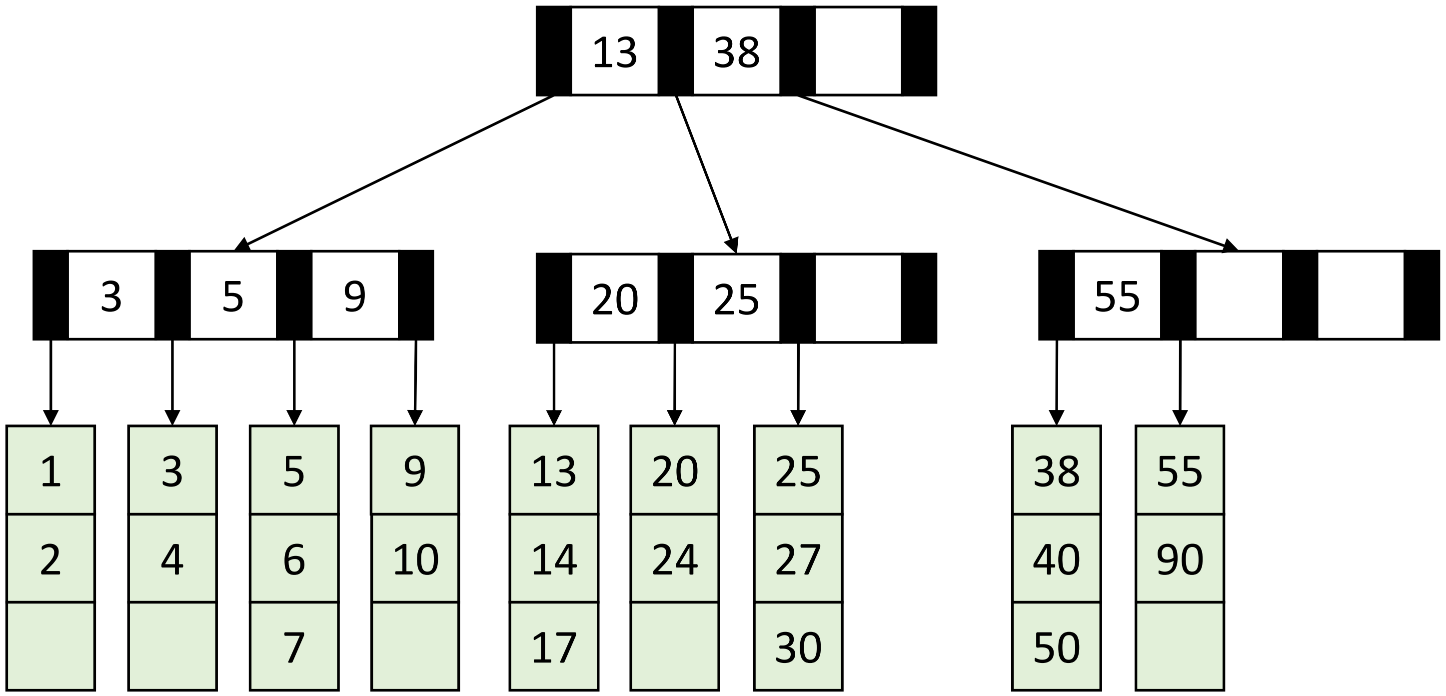
Insert Example

Insert 26



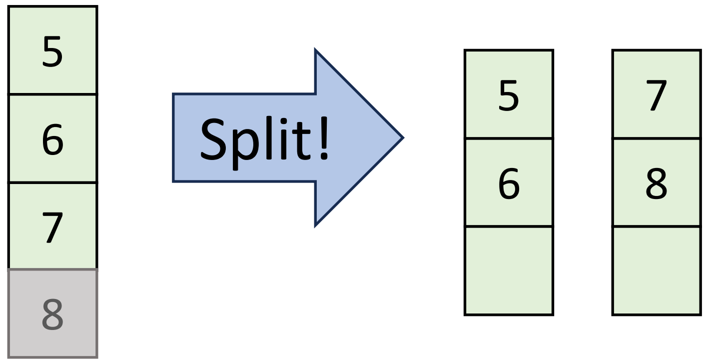
Insert Example

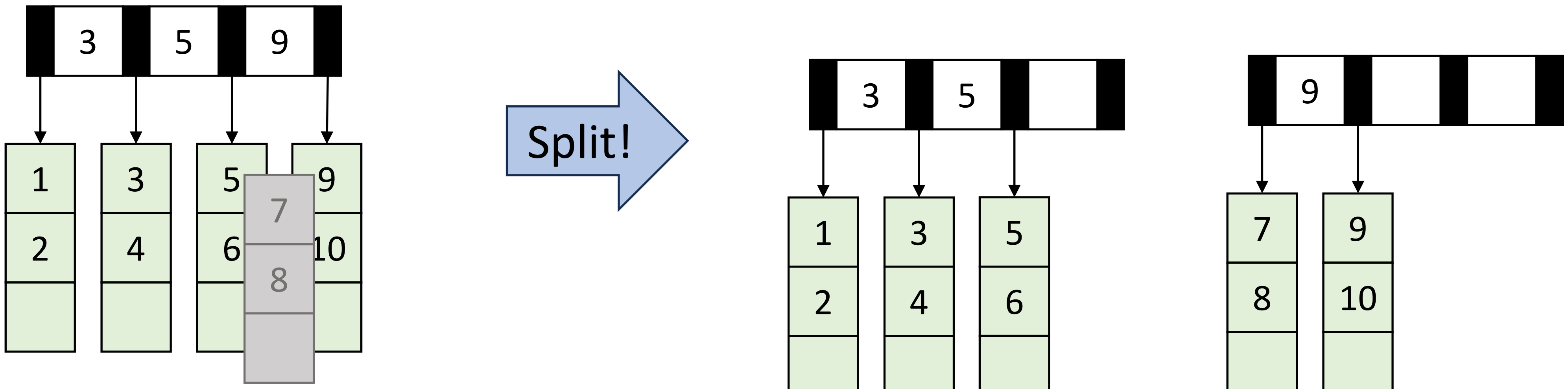
Insert 8



Insert Example

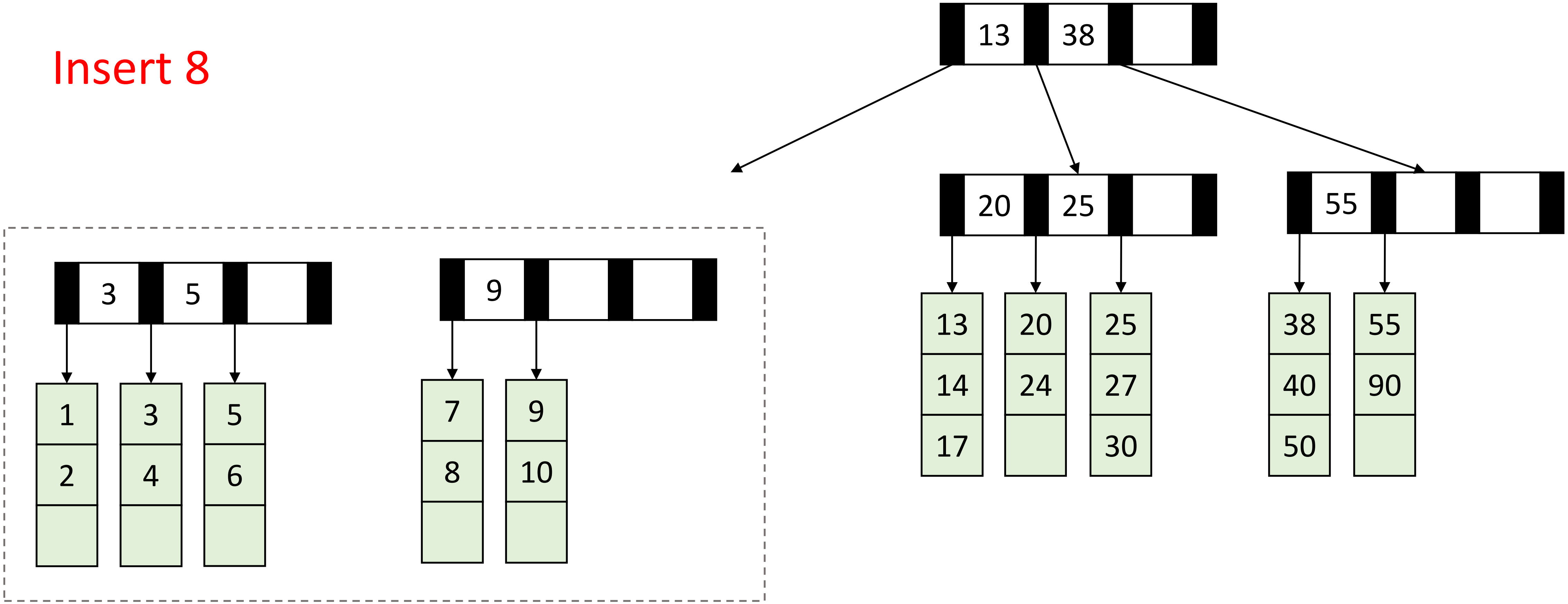
Insert 8





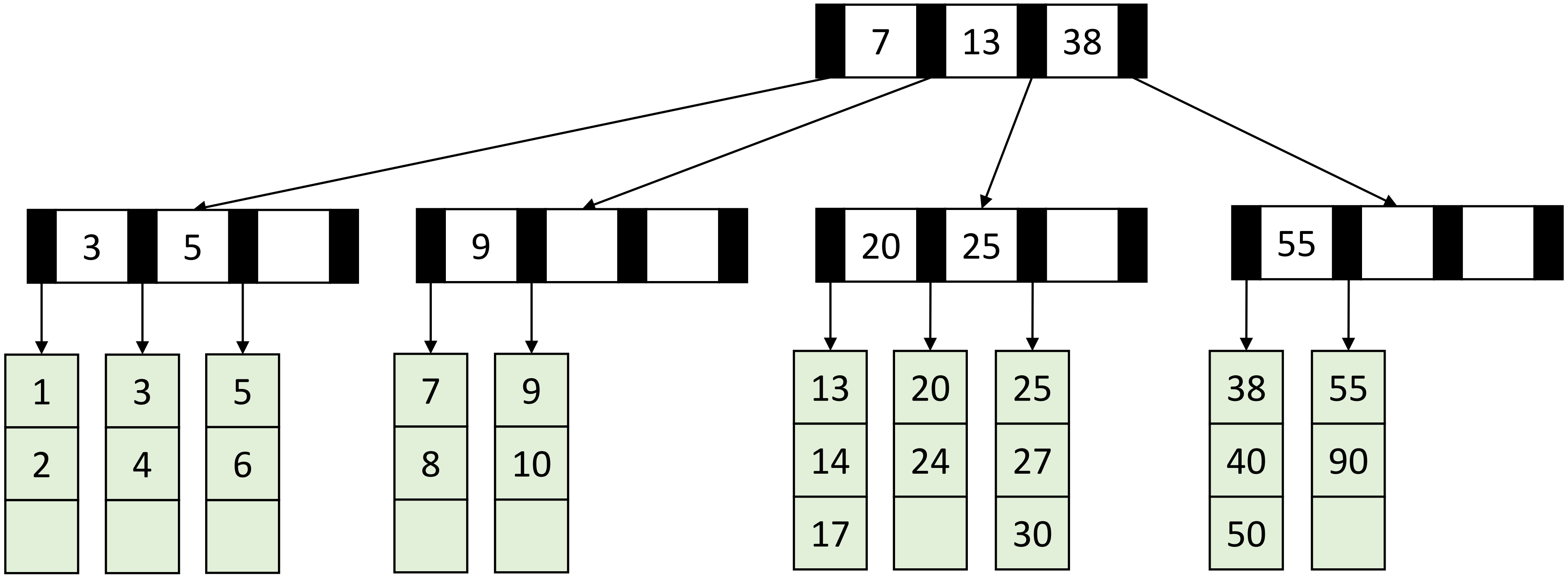
Insert Example

Insert 8



Insert Example

Insert 8



Let’s do it together!

- $M = 3, L = 3$
- Insert all of these:

Running Time of Find

- Maximum number of leaves:
  - $\frac{2n}{L}$
  - $\Theta\left(\frac{n}{L}\right)$
- Maximum height of the tree:
  - $2\log_M \frac{2n}{L}$
  - $\Theta\left(\log_M \frac{n}{L}\right)$
- Find:
  - One binary search per level of the tree
    - $\Theta(\log_2 M)$ per search
  - One binary search in the leaf
    - $\Theta(\log_2 L)$

Overall: $\Theta\left(\log_2 M \cdot \log_M \frac{n}{L} + \log_2 L\right)$

Usually simplified to:

$$\Theta(\log_2 M \cdot \log_M n)$$

Running Time of Insert

- Find:
  - $\Theta(\log_2 M \cdot \log_M n)$
- Add item to leaf:
  - $\Theta(L)$
- Split a leaf
  - $\Theta(L)$
- Split one internal node:
  - $\Theta(M)$

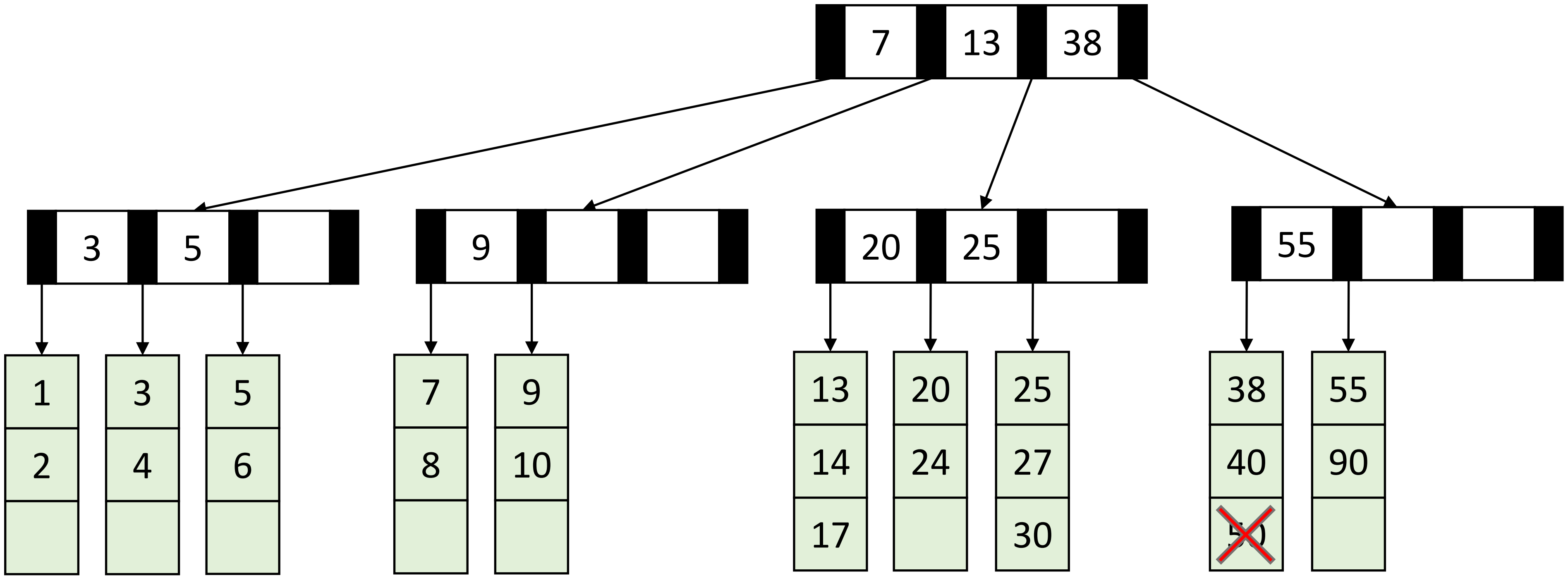
Overall: $\Theta(L + M \cdot \log_M n)$
Usually simplified to:

$$\Theta(\log_2 M \cdot \log_M n)$$

Delete

- Recall: all nodes must be at least half full (except root at startup)

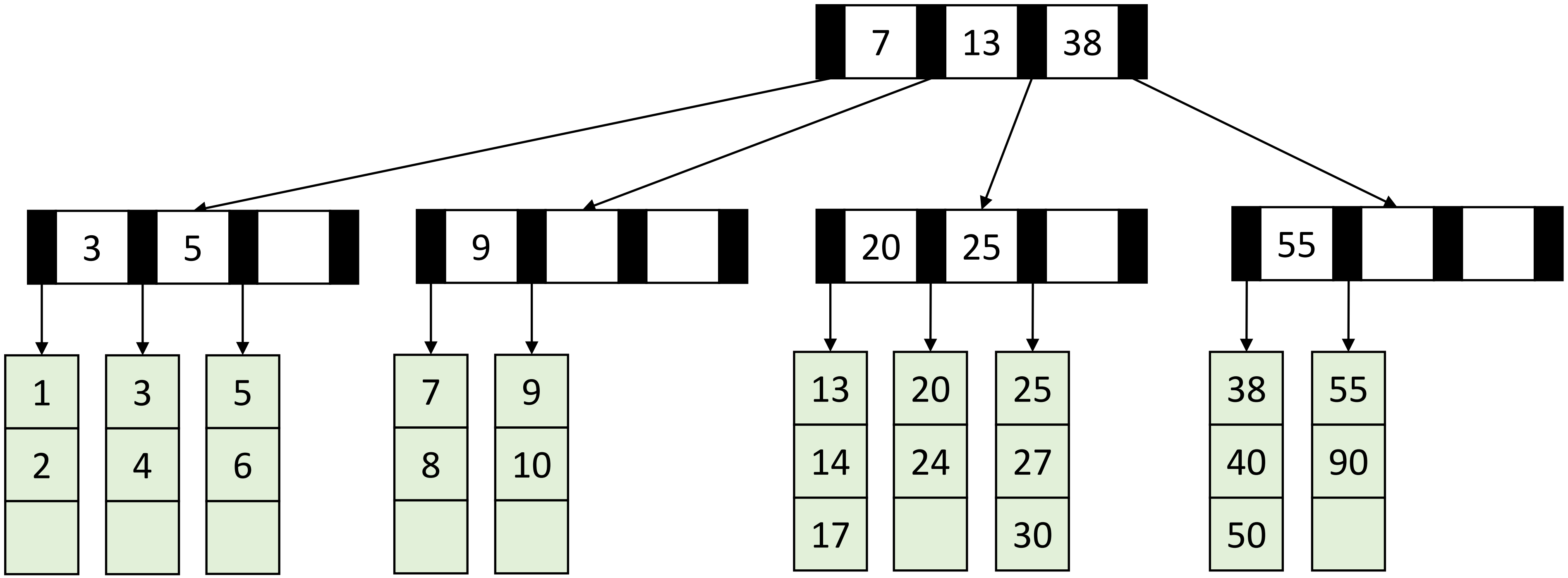
delete 50



Delete

- Recall: all nodes must be at least half full (except root at startup)

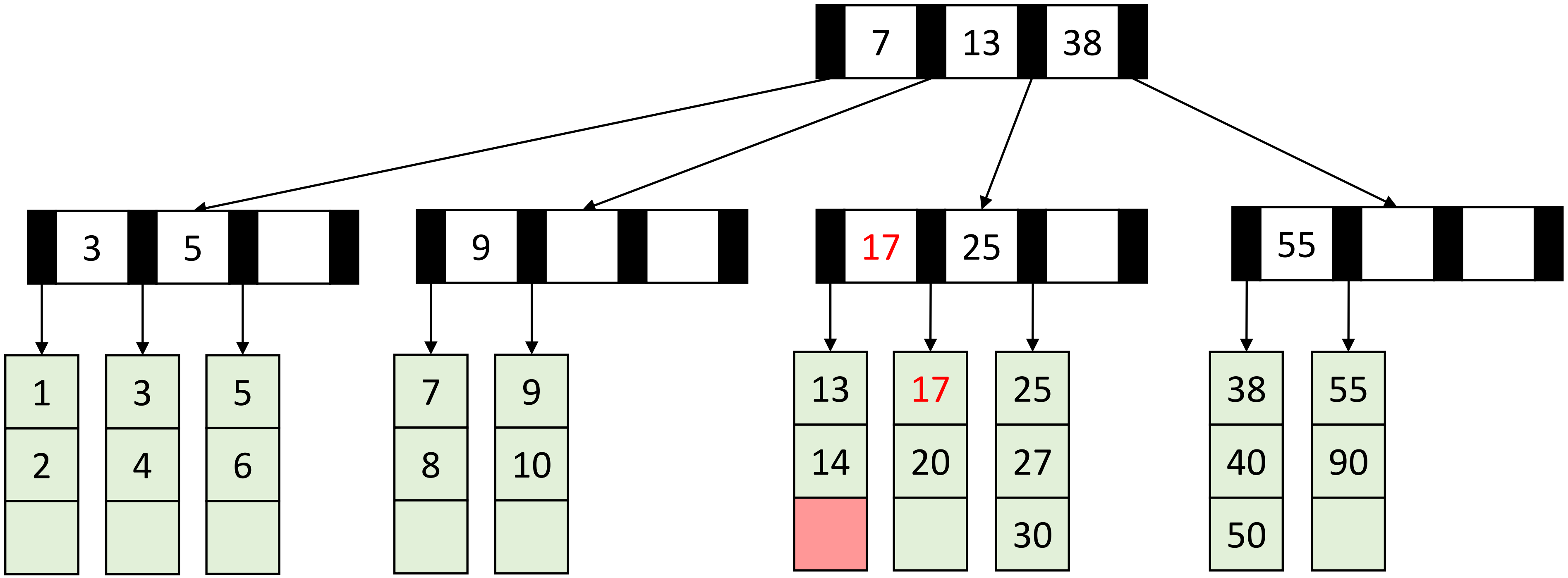
delete 24



Delete

- Recall: all nodes must be at least half full (except root at startup)

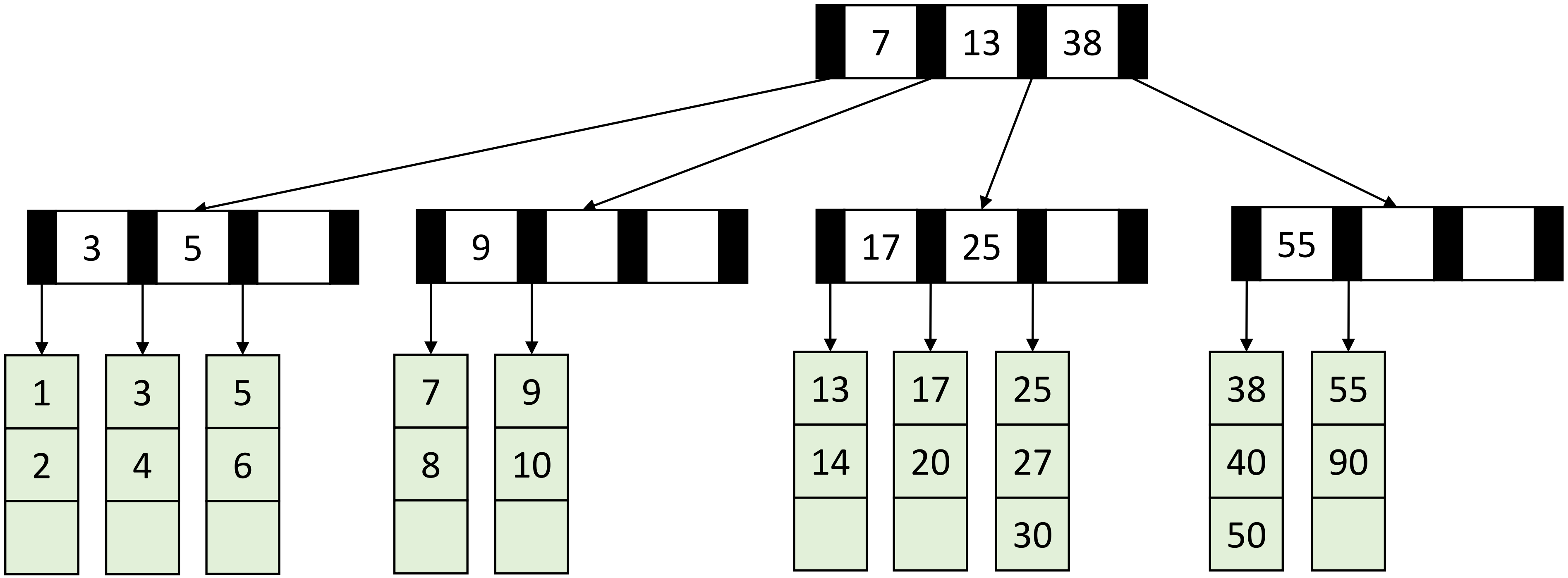
delete 24



Delete

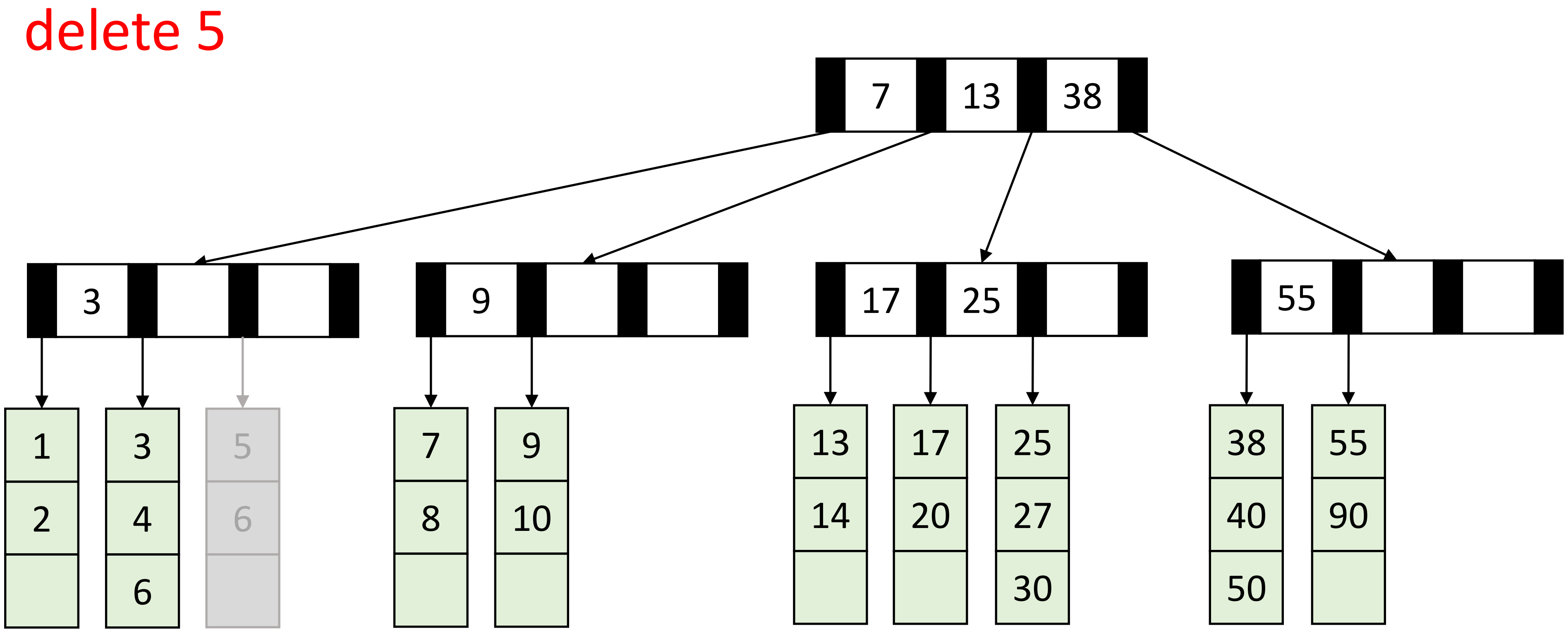
- Recall: all nodes must be at least half full (except root at startup)

delete 5



Delete

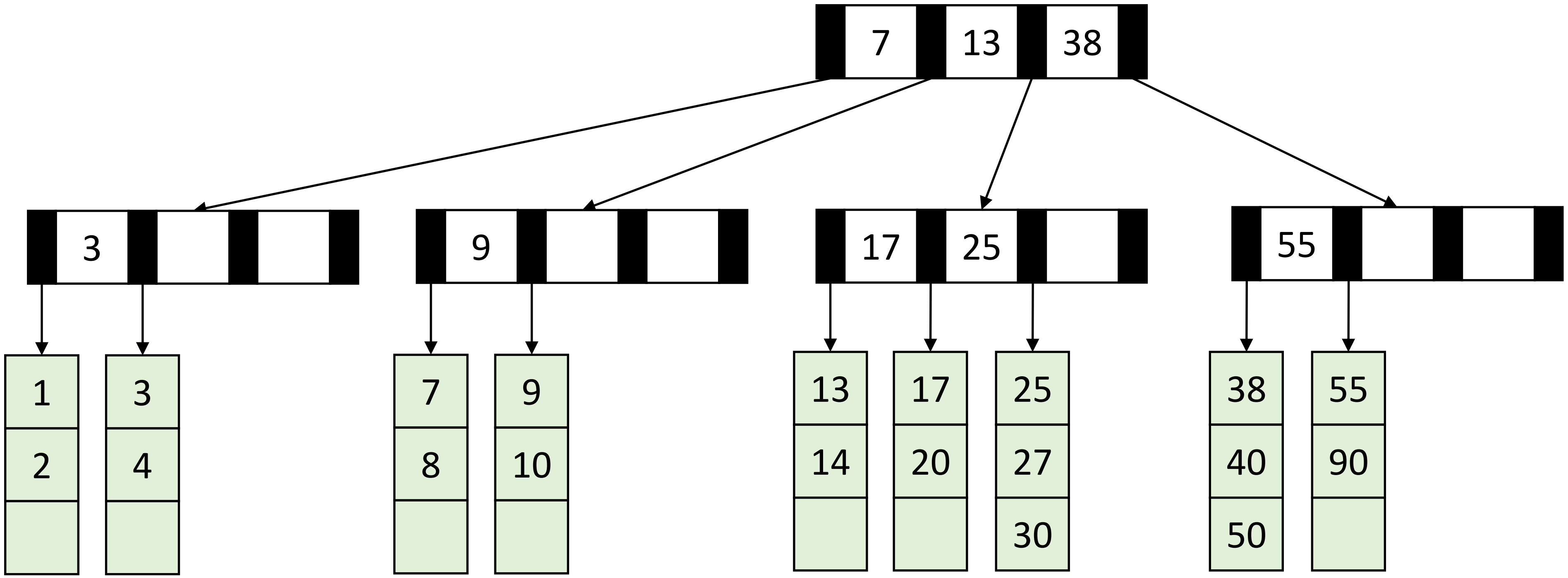
- Recall: all nodes must be at least half full (except root at startup)



Delete

- Recall: all nodes must be at least half full (except root at startup)

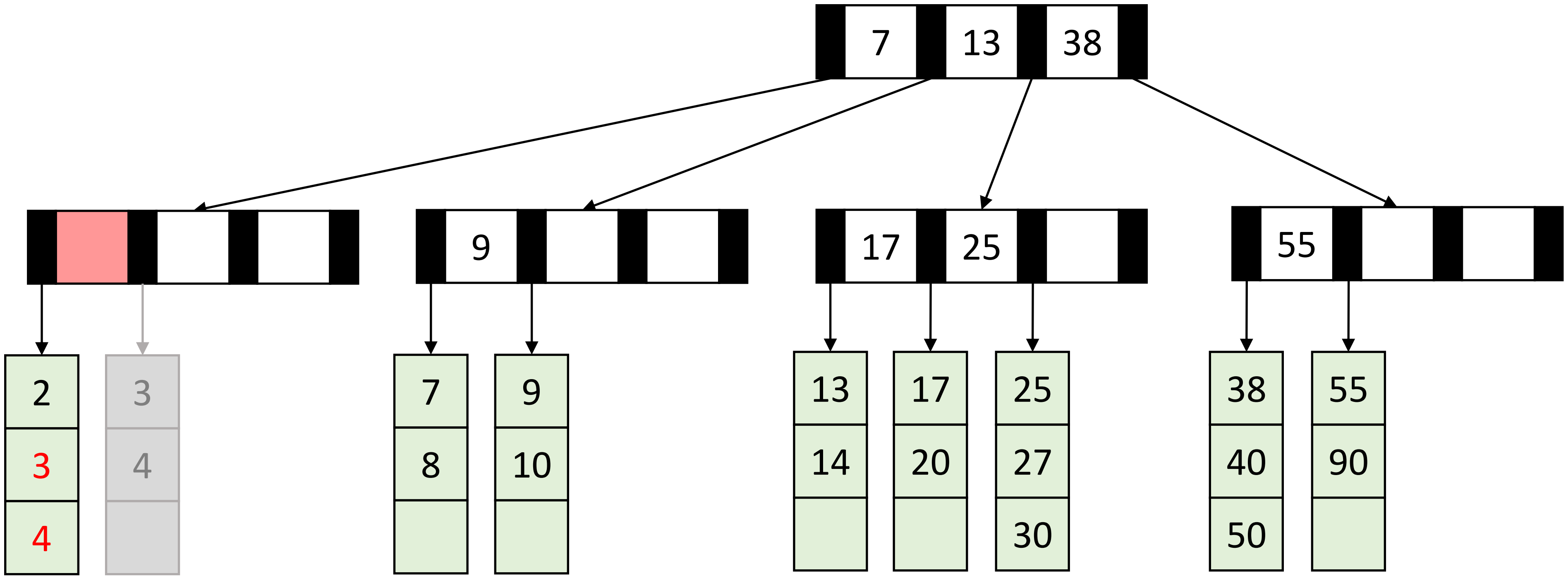
delete 1



Delete

- Recall: all nodes must be at least half full (except root at startup)

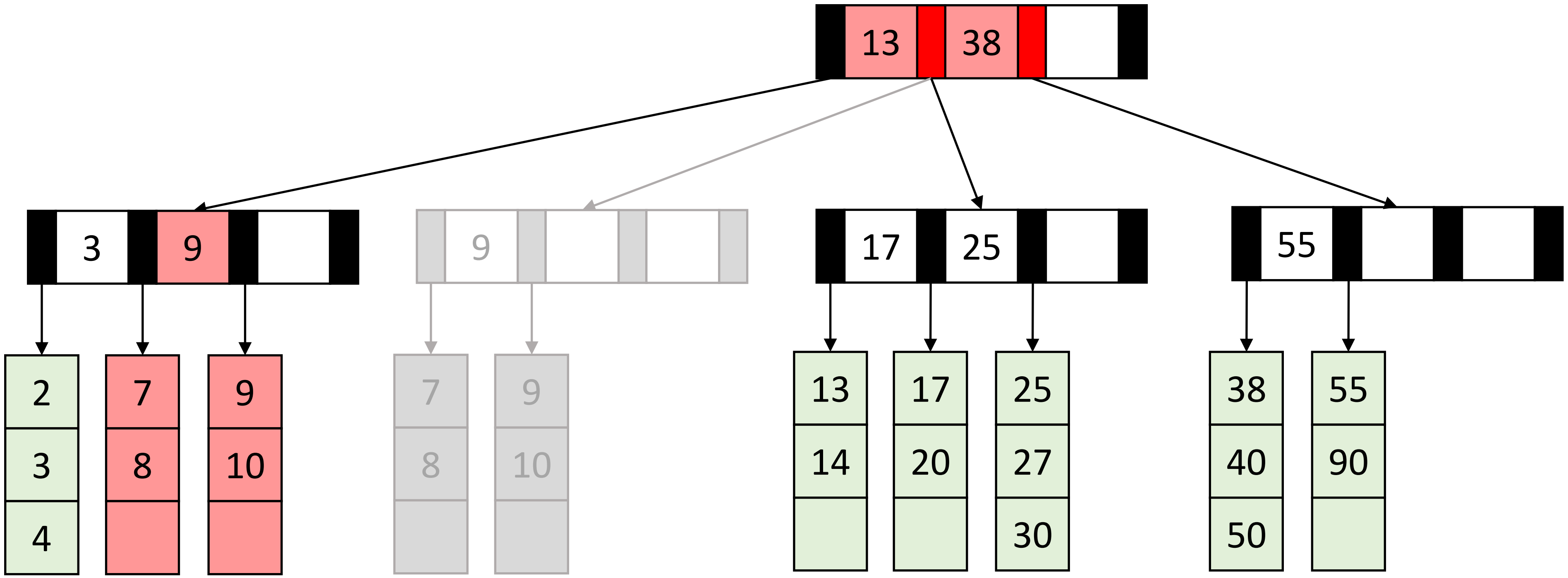
delete 1



Delete

- Recall: all nodes must be at least half full (except root at startup)

delete 1



Delete Summary

- Find the item
- Remove the item from the leaf
    - If that causes the leaf to be under-full, adopt from a neighbor
    - If that would cause the neighbor to be under-full, merge those two leaves
    - Update the parent
        - If that causes the parent to be under-full, adopt from a neighbor
        - If that causes the neighbor to be under-full, merge
        - Update the parent
            - …

Delete TLDR

- Find and remove from leaf
- Keep doing this until everything is “full enough”:
  - If the node is now too small, adopt from a neighbor
  - If the neighbor is too small then merge

Next topic: Hash Tables

Data Structure	Time to insert	Time to find	Time to delete
Unsorted Array	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Unsorted Linked List	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Sorted Array	$\Theta(n)$	$\Theta(\log n)$	$\Theta(n)$
Sorted Linked List	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Binary Search Tree	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
AVL Tree	$\Theta(\log n)$	$\Theta(\log n)$	$\Theta(\log n)$
Hash Table (Worst case)	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
Hash Table (Average)	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$

Two Different ideas of “Average”

- Expected Time
  - The expected number of operations a randomly-chosen input uses
  - Assumed randomness from somewhere
    - Most simply: from the input
    - Preferably: from the algorithm/data structure itself
  - $f(n) =$ sum of the running times for each input of size n divided by the number of inputs of size n
- Amortized Time
  - The long-term average per-execution cost (in the worst case)
  - Rather than look at the worst case of one execution, look at the total worst case of a sequential chain of many executions
    - Why? The worst case may be guaranteed to be rare
  - $f(n) =$ the sum of the running times from a sequence of n sequential calls to the function divided by n

Amortized Example

- ArrayList Insert:
  - Worst case: $\Theta(n)$

0	1	2	3	4	5	6	7	8

0	1	2	3	4	5	6	7	8							

Amortized Example

- ArrayList Insert:
  - First 8 inserts: 1 operation each
  - 9^{th} insert: 9 operations
  - Next 7 inserts: 1 operation each
  - 17^{th} insert: 17 operations
  - Next 15 inserts: 1 operation each
  - …

Do x operations with cost 1
Do 1 operation with cost x
Do x operations with cost 1
Do 1 operation with cost $2x$
Do $2x$ operations with cost 1
Do 1 operation with cost $4x$
Do $4x$ operations with cost 1
Do 1 operation with cost $8x$
…
Amortized: each operation cost 2 operations
$\Theta(1)$

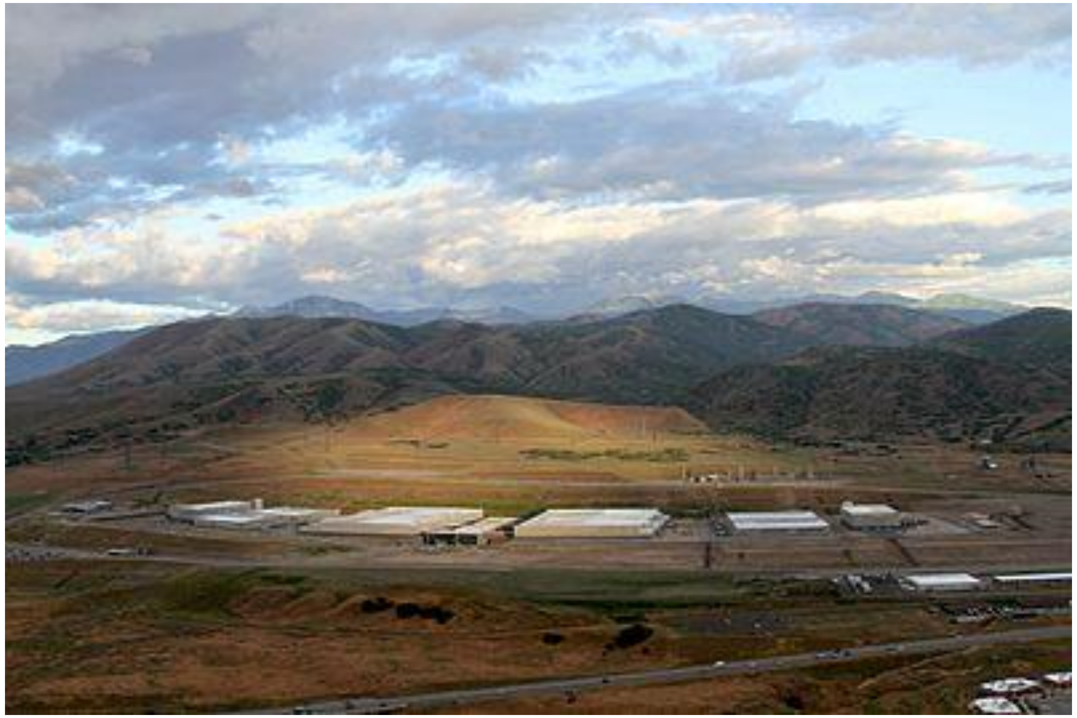
0	1	2	3	4	5	6	7	8

0	1	2	3	4	5	6	7	8							

Hash Tables

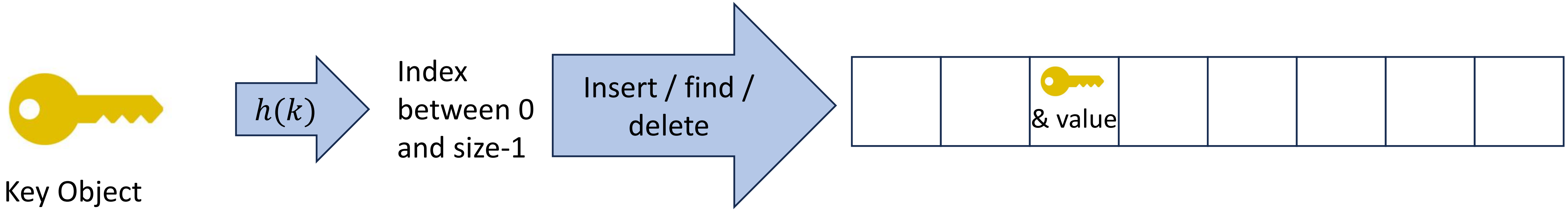
- Motivation:
  - Why not just have a gigantic array?

Problem?



Hash Tables

- Idea:
  - Have a small array to store information
  - Use a **hash function** to convert the key into an index
    - Hash function should "scatter" the keys, behave as if it randomly assigned keys to indices
  - Store key at the index given by the hash function
  - Do something if two keys map to the same place (should be very rare)
    - Collision resolution



Example

0	1	2	3	4	5	6	7	8	9

- Key: Phone Number
- Value: People
- Table size: 10
- $h(phone) =$ number as an integer % 10
- $h(8675309) = 9$

What Influences Running time?

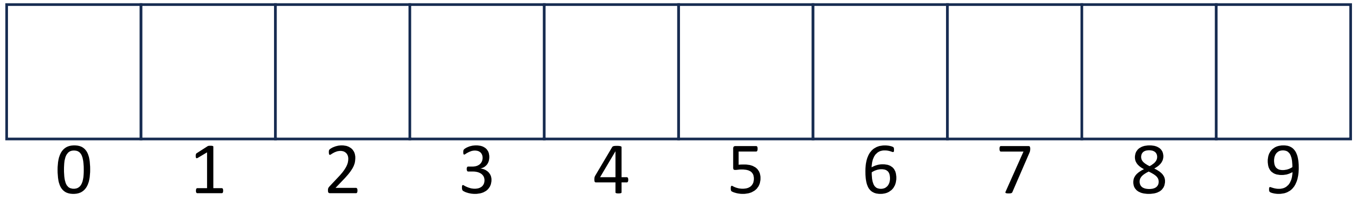
Properties of a “Good” Hash

- Definition: A hash function maps objects to integers

- Should be very efficient
  - Calculating the hash should be negligible
- Should randomly scatter objects
  - Objects that are similar to each other should be likely to end up far away
- Should use the entire table
  - There should not be any indices in the table that nothing can hash to
  - Picking a table size that is prime helps with this
- Should use things needed to “identify” the object
  - Use only fields you would check for a .equals method be included in calculating the hash
  - More fields typically leads to fewer collisions, but less efficient calculation

A Bad Hash (and phone number trivia)

- $h(phone) =$ the first digit of the phone number
  - No US phone numbers start with 1 or 0
  - If we’re sampling from this class, 2 is by far the most likely

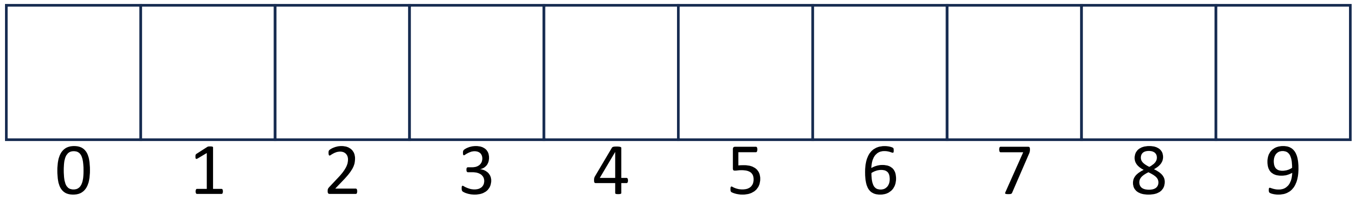


Compare These Hash Functions (for strings)

- Let $s = s_0 s_1 s_2 \dots s_{m-1}$ be a string of length m
  - Let $a(s_i)$ be the ascii encoding of the character s_i
- $h_1(s) = a(s_0)$
- $h_2(s) = \left(\sum_{i=0}^{m-1} a(s_i)\right)$
- $h_3(s) = \left(\sum_{i=0}^{m-1} a(s_i) \cdot 37^i\right)$

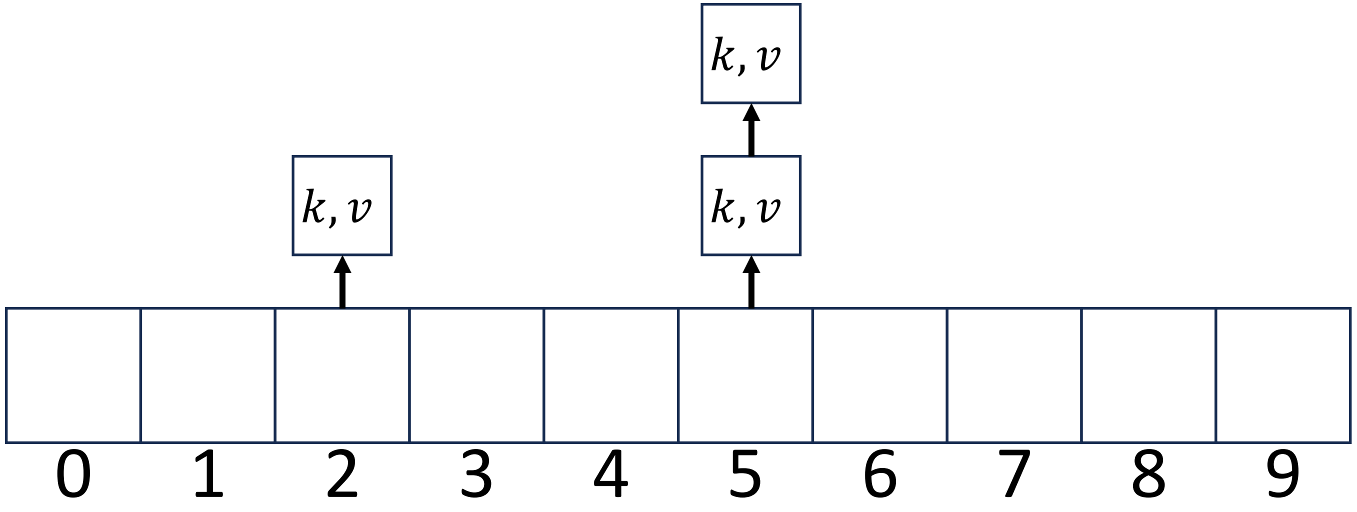
Collision Resolution

- A Collision occurs when we want to insert something into an already-occupied position in the hash table
- 2 main strategies:
  - Separate Chaining
    - Use a secondary data structure to contain the items
      - E.g. each index in the hash table is itself a linked list
  - Open Addressing
    - Use a different spot in the table instead
      - Linear Probing
      - Quadratic Probing
      - Double Hashing



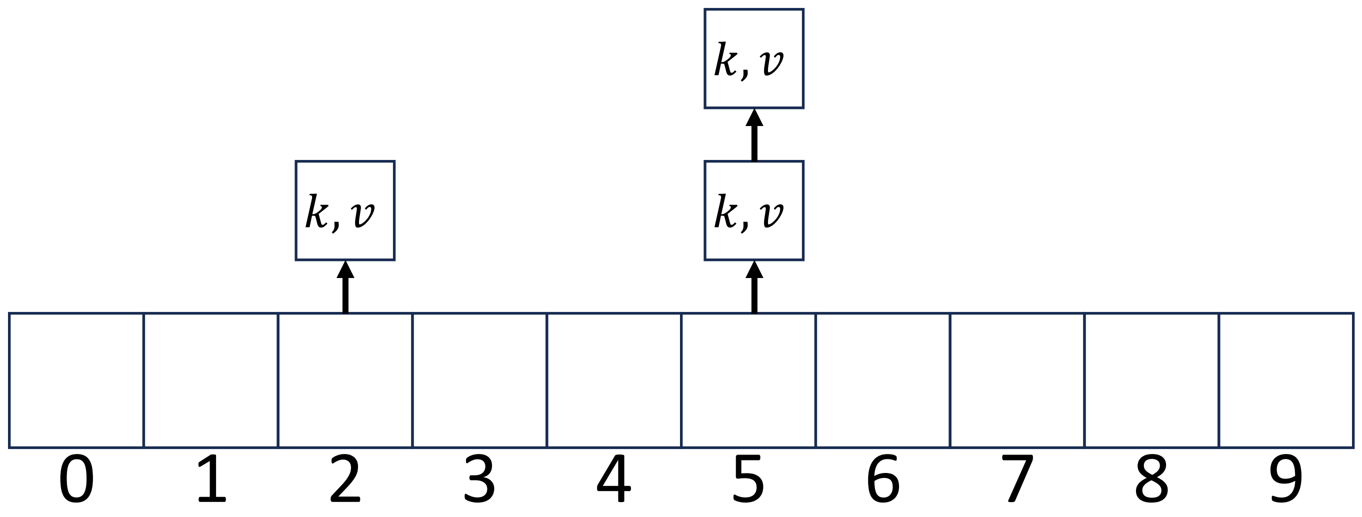
Separate Chaining Insert

- To insert k, v:
  - Compute the index using $i = h(k)\ \%\ size$
  - Add the key-value pair to the data structure at $table[i]$



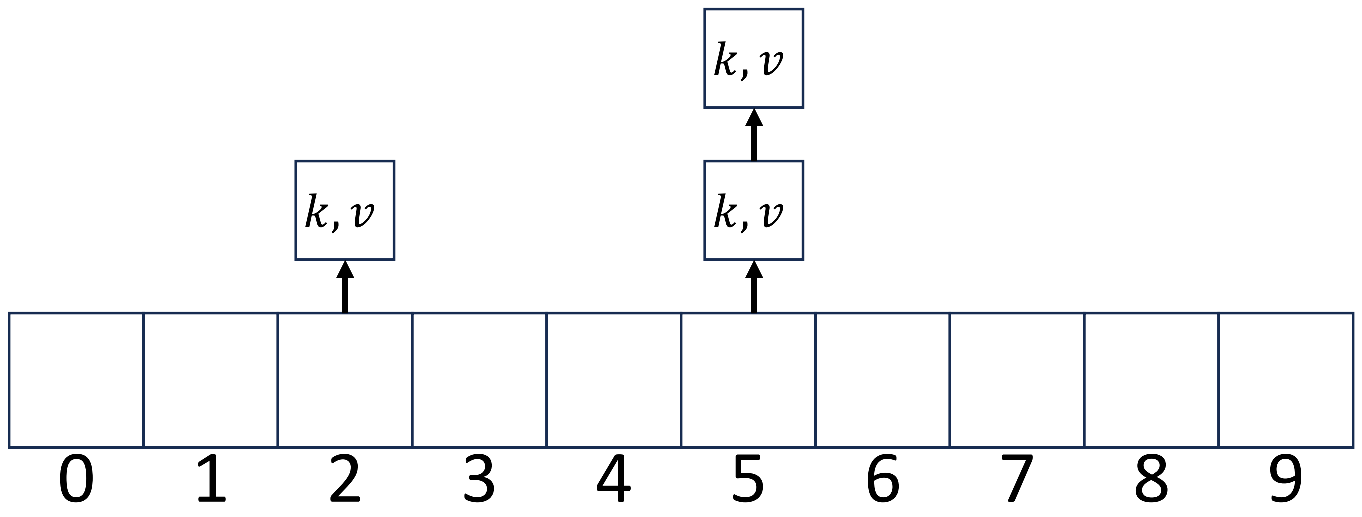
Separate Chaining Find

- To find k:
  - Compute the index using $i = h(k)\ \%\ size$
  - Call find with the key on the data structure at $table[i]$



Separate Chaining Delete

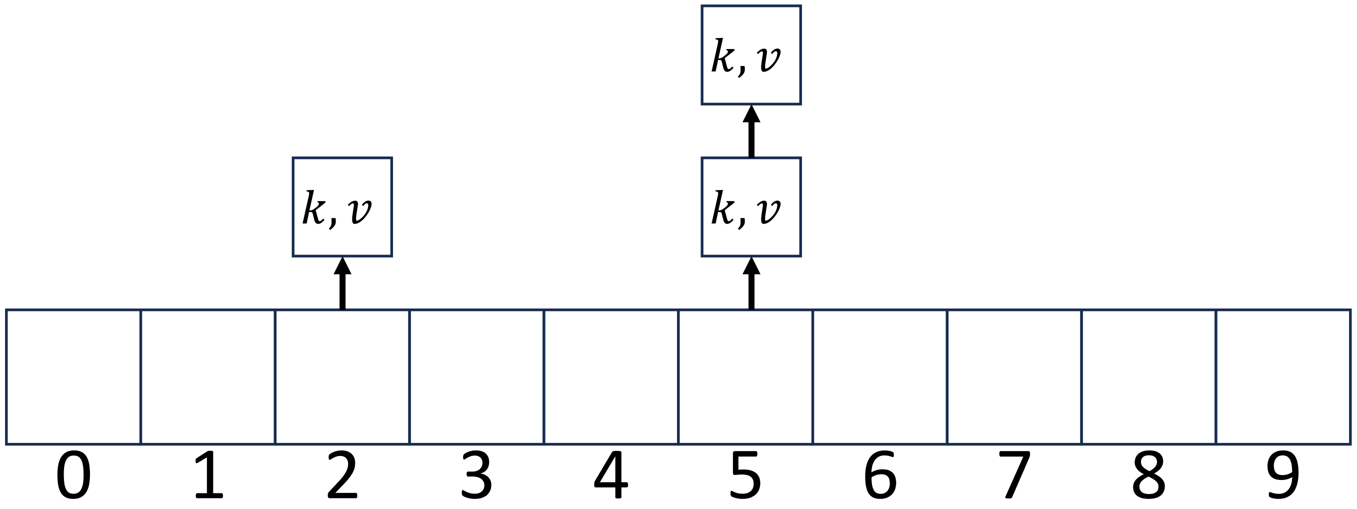
- To delete k:
  - Compute the index using $i = h(k)\ \%\ size$
  - Call delete with the key on the data structure at $table[i]$



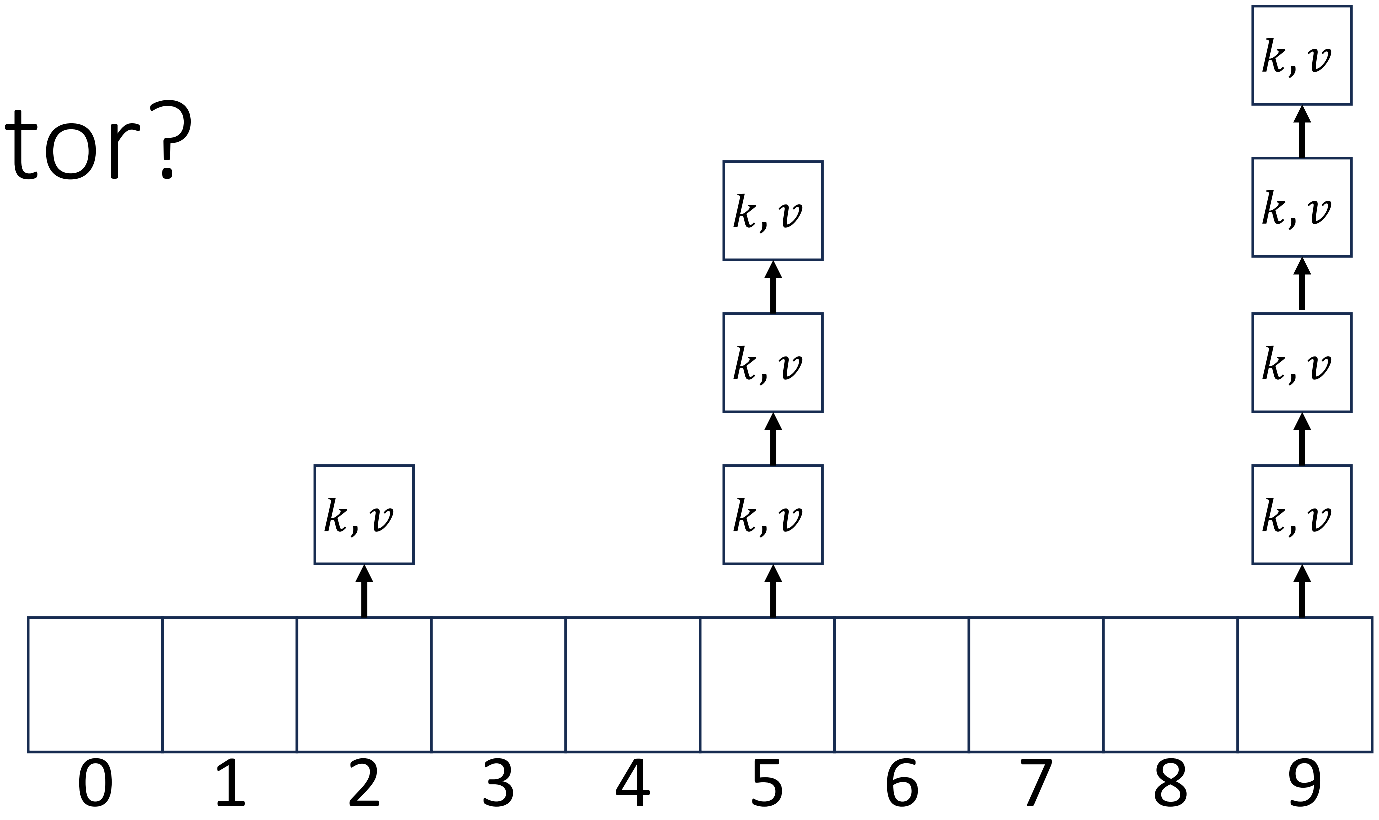
Formal Running Time Analysis

- The **load factor** of a hash table represents the average number of items per “bucket”
    - $\lambda = \frac{n}{size}$
- Assume we have a has table that uses a linked-list for separate chaining
    - What is the expected number of comparisons needed in an unsuccessful find?
    - What is the expected number of comparisons needed in a successful find?
- How can we make the expected running time $\Theta(1)$?

Load Factor?



Load Factor?



Load Factor?