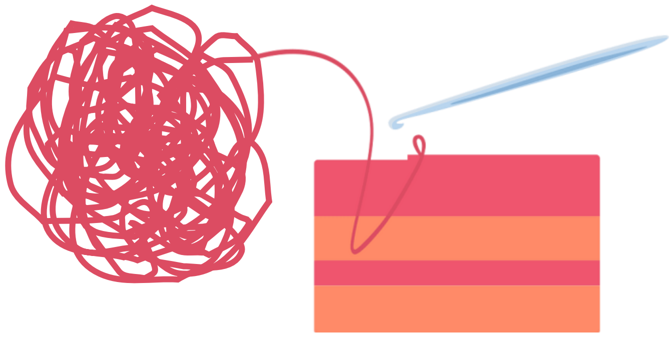
CSE 332 Autumn 2023 Lecture 2: Algorithm Analysis pt.2

Nathan Brunelle

http://www.cs.uw.edu/332

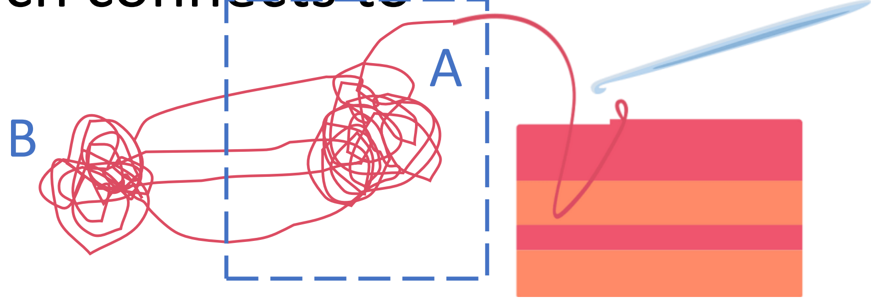
End-of-Yarn Finding



1. Set aside the already-obtained "beginning"

2. If you see the end of the yarn, you're done!

3. Separate the pile of yarn into 2 piles, note which connects to the beginning (call it pile A, the other pile B)



Repeat on pile with end

4. Count the number of strands crossing the piles

5. If the count is even, pile A contains the end, else pile B does

Running Time Analysis

- Units of “time”
- How do we express running time?

Why Do resource Analysis?

- Allows us to compare *algorithms,* not implementations
  - Using observations necessarily couples the algorithm with its implementation
  - If my implementation on my computer takes more time than your implementation on your computer, we cannot conclude your algorithm is better
- We can predict an algorithm’s running time before implementing
- Understand where the bottlenecks are in our algorithm

Goals for Algorithm Analysis

- Identify a *function* which maps the algorithm's input size to a measure of resources used
  - Domain of the function: **sizes** of the input
    - Number of characters in a string, number of items in a list, number of pixels in an image
  - Codomain of the function: **counts** of resources used
    - Number of times the algorithm adds two numbers together, number times the algorithm does a > or < comparison, maximum number of bytes of memory the algorithm uses at any time
- Important note: Make sure you know the "units" of your domain and codomain!

Worst Case Analysis (in general)

- If an algorithm has a worst case resource complexity of $f(n)$
  - Among all possible size-n inputs, the “worst” one will use $f(n)$ “resources”
  - I.e. $f(n)$ gives the maximum count of resources needed from among all inputs of size n

Worst Case Running Time Analysis

- If an algorithm has a worst case running time of $f(n)$
  - Among all possible size-n inputs, the “worst” one will do $f(n)$ “operations”
  - I.e. $f(n)$ gives the maximum operation count from among all inputs of size n

Worst Case Space Analysis

- If an algorithm has a worst case space complexity of $f(n)$
  - Among all possible size-n inputs, the "worst" one will need $f(n)$ "memory units"
  - I.e. $f(n)$ gives the maximum memory unit count from among all inputs of size n

Worst Case Running Time - Example

```
myFunction(List n){
    b = 55 + 5;
    c = b / 3;
    b = c + 100;
    for (i = 0; i < n.size(); i++) {
        b++;
    }
    if (b % 2 == 0) {
        c++;
    }
    else {
        for (i = 0; i < n.size(); i++) {
            c++;
        }
    }
    return c;
}
```

Questions to ask:

- What are the units of the input size?
- What are the operations we're counting?
- For each line:
    - How many times will it run?
    - How long does it take to run?
    - Does this change with the input size?

Worst Case Running Time – Example 2

```
beAnnoying(List n){
    List m = [];
    for (i=0; i < n.size(); i++){
        m.add(n[i]);
        for (j=0; j< n.size(); j++){
            print (“Hi, I’m annoying”);
        }
    }
    return;
}
```

Questions to ask:

- What are the units of the input size?
- What are the operations we’re counting?
- For each line:
    - How many times will it run?
    - How long does it take to run?
    - Does this change with the input size?

Worst Case Running Time – General Guide

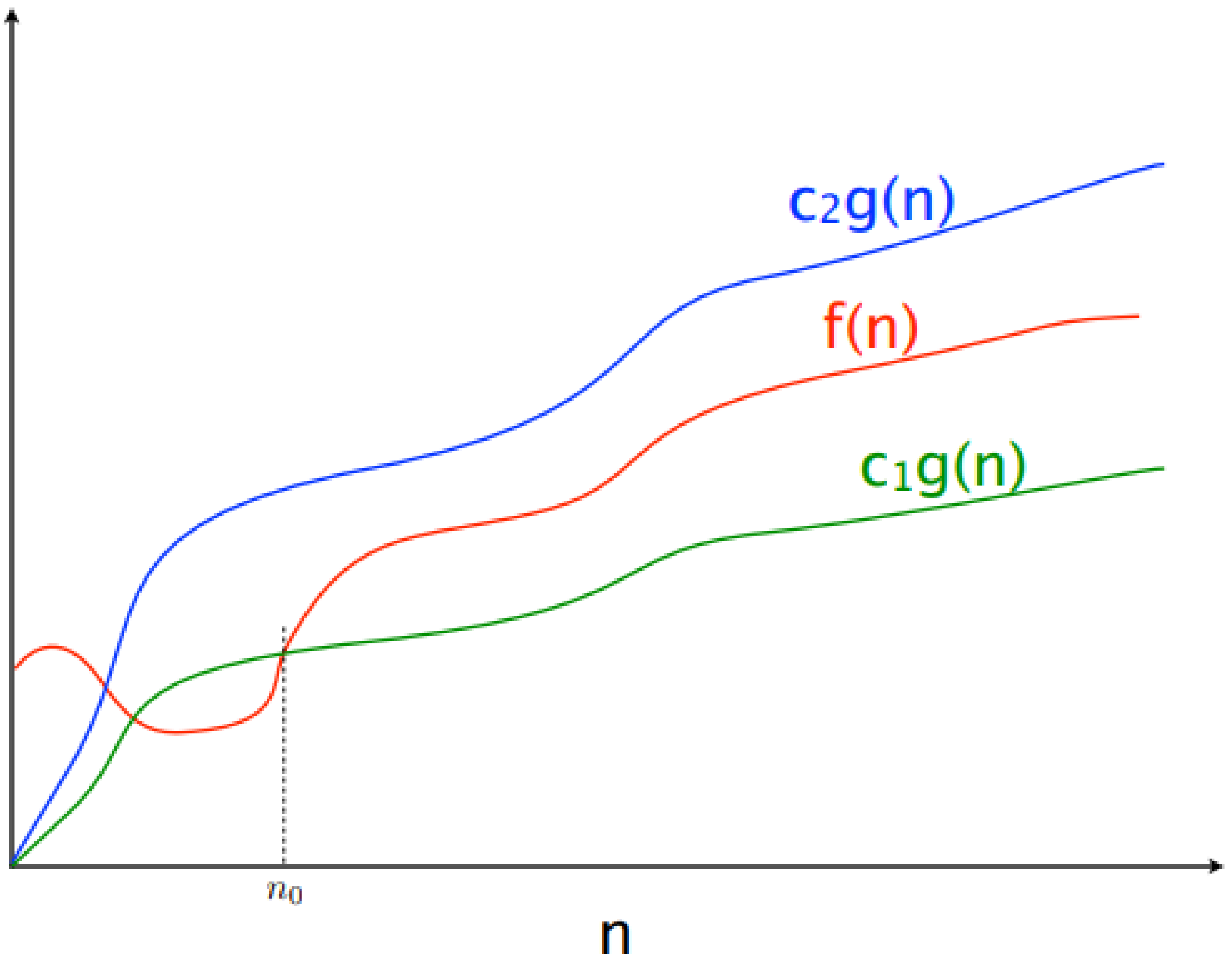
- Add together the time of consecutive statements
- Loops: Sum up the time required through each iteration of the loop
  - If each takes the same time, then [time per loop × number of iterations]
- Conditionals: Sum together the time to check the condition and time of the slowest branch
- Function Calls: Time of the function's body
- Recursion: Solve a recurrence relation

Searching in a Sorted List

Binary Search

Comparing Running Times

- Suppose I have these algorithms, all of which have the same input/output behavior:
  - Algorithm A's worst case running time is $10n + 900$
  - Algorithm B's worst case running time is $100n - 50$
  - Algorithm C's worst case running time is $\frac{n^2}{100}$
- Which algorithm is best?



$$f(n) = O(g(n))$$

$$f(n) = \Theta(g(n))$$

$$f(n) = \Omega(g(n))$$

Asymptotic Notation

- $O\big(g(n)\big)$
  - The **set of functions** with asymptotic behavior less than or equal to $g(n)$
  - Upper-bounded by a constant times g for large enough values n
  - $f \in O\big(g(n)\big) \equiv \exists c > 0. \exists n_0 > 0. \forall n \geq n_0. f(n) \leq c \cdot g(n)$
- $\Omega(g(n))$
  - the **set of functions** with asymptotic behavior greater than or equal to $g(n)$
  - Lower-bounded by a constant times g for large enough values n
  - $f \in \Omega\big(g(n)\big) \equiv \exists c > 0. \exists n_0 > 0. \forall n \geq n_0. f(n) \geq c \cdot g(n)$
- $\Theta\big(g(n)\big)$
  - "Tightly" within constant of g for large n
  - $\Omega\big(g(n)\big) \cap O(g(n))$

Asymptotic Notation Example

- Show: $10n + 100 \in O(n^2)$
  - **Technique:** find values $c > 0$ and $n_0 > 0$ such that $\forall n > n_0 . \, 10n + 100 \leq c \cdot n^2$
  - **Proof:**

Asymptotic Notation Example

- Show: $10n + 100 \in O(n^2)$
  - **Technique:** find values $c > 0$ and $n_0 > 0$ such that $\forall n \geq n_0.\, 10n + 100 \leq c \cdot n^2$
  - **Proof:** Let $c = 10$ and $n_0 = 6$. Show $\forall n \geq 6. 10n + 100 \leq 10n^2$

$$10n + 100 \leq 10n^2$$
$$\equiv n + 10 \leq n^2$$
$$\equiv 10 \leq n^2 - n$$
$$\equiv 10 \leq n(n-1)$$

This is True because $n(n-1)$ is strictly increasing and $6(6-1) > 10$

Asymptotic Notation Example

- Show: $13\mathrm{n}^2 - 50\mathrm{n} \in \Omega(n^2)$
  - **Technique:** find values $c > 0$ and $n_0 > 0$ such that $\forall n \geq n_0 .\ 13n^2 - 50n \geq c \cdot n^2$
  - **Proof:**

Asymptotic Notation Example

- Show: $13\mathrm{n}^2 - 50\mathrm{n} \in \Omega(n^2)$
  - **Technique:** find values $c > 0$ and $n_0 > 0$ such that $\forall n \geq n_0.\, 13n^2 - 50n \geq c \cdot n^2$
  - **Proof:** let $c = 12$ and $n_0 = 50$. Show $\forall n \geq 50.\, 13n^2 - 50n \geq 12n^2$

$$13n^2 - 50n \geq 12n^2$$
$$\equiv n^2 - 50n \geq 0$$
$$\equiv n^2 \geq 50n$$
$$\equiv n \geq 50$$

This is certainly true $\forall n \geq 50$.

Asymptotic Notation Example

- Show: $n^2 \notin O(n)$

Asymptotic Notation Example

Proof by Contradiction!

- To Show: $n^2 \notin O(n)$
  - **Technique: Contradiction**
  - **Proof:** Assume $n^2 \in O(n)$. Then $\exists c, n_0 > 0$ s.t. $\forall n > n_0, n^2 \leq cn$
    Let us derive constant c. For all $n > n_0 > 0$, we know:
    $cn \geq n^2$,
    $c \geq n$.

    Since c is lower bounded by n, c cannot be a constant and make this True.
    Contradiction. Therefore $n^2 \notin O(n)$.

Gaining Intuition

- When doing asymptotic analysis of functions:
  - If multiple expressions are added together, ignore all but the "biggest"
    - If $f(n)$ grows asymptotically faster than $g(n)$, then $f(n) + g(n) \in \Theta\big(f(n)\big)$
  - Ignore all multiplicative constants
    - $f(n) + c \in \Theta\big(f(n)\big)$ for any constant $c \in \mathbb{R}$
  - Ignore bases of logarithms
  - Do NOT ignore:
    - Non-multiplicative and non-additive constants (e.g. in exponents, bases of exponents)
    - Logarithms themselves
- Examples:
  - $4n + 5$
  - $0.5n\log n + 2n + 7$
  - $n^3 + 2^n + 3n$
  - $n\log(10n^2)$

More Examples

- Is each of the following True or False?
  - $4 + 3n \in \ O(n)$
  - $n + 2\log n \in \ O(\log n)$
  - $\log n + 2 \in \ O(1)$
  - $n^{50} \in \ O(1.1^n)$
  - $3^n \in \Theta(2^n)$