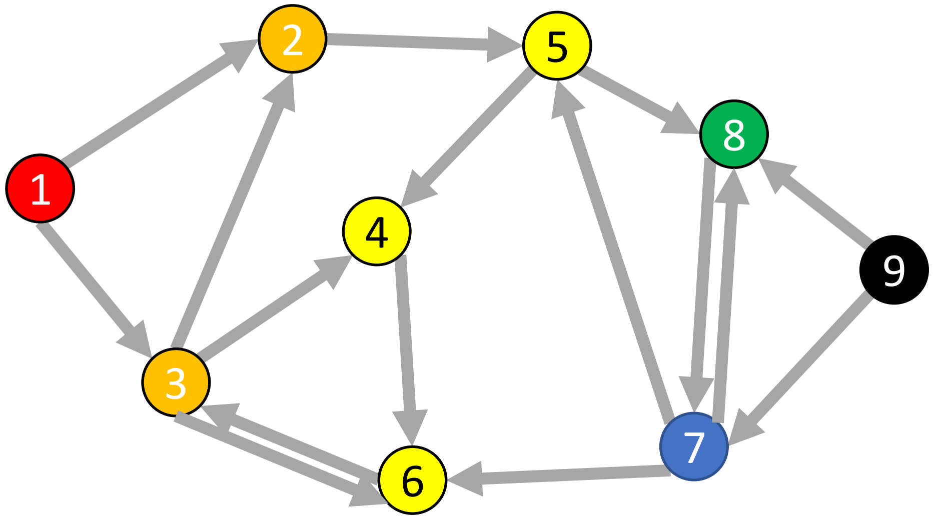
CSE 332 Autumn 2023
Lecture 21: Dijkstra’s

Nathan Brunelle

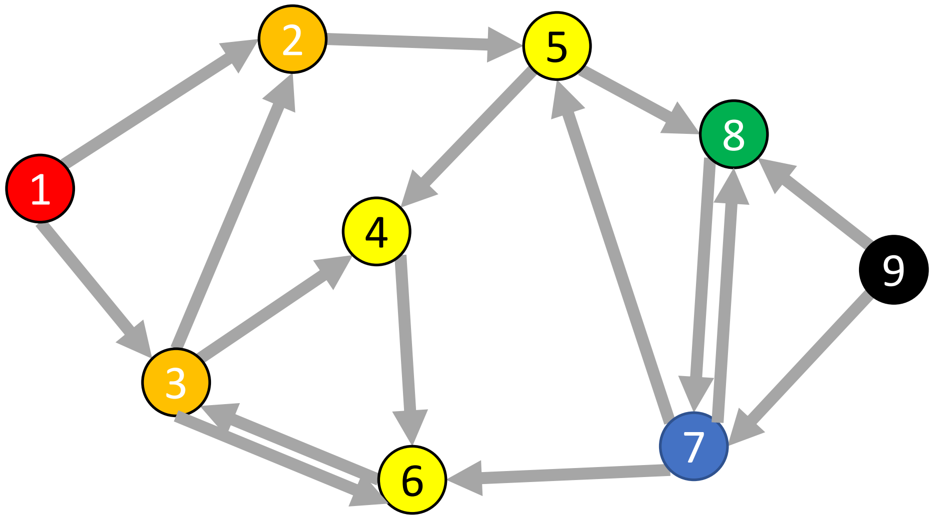
http://www.cs.uw.edu/332

Breadth-First Search

- Input: a node s
- Behavior: Start with node s, visit all neighbors of s, then all neighbors of neighbors of s, …
- Output:
  - How long is the shortest path?
  - Is the graph connected?



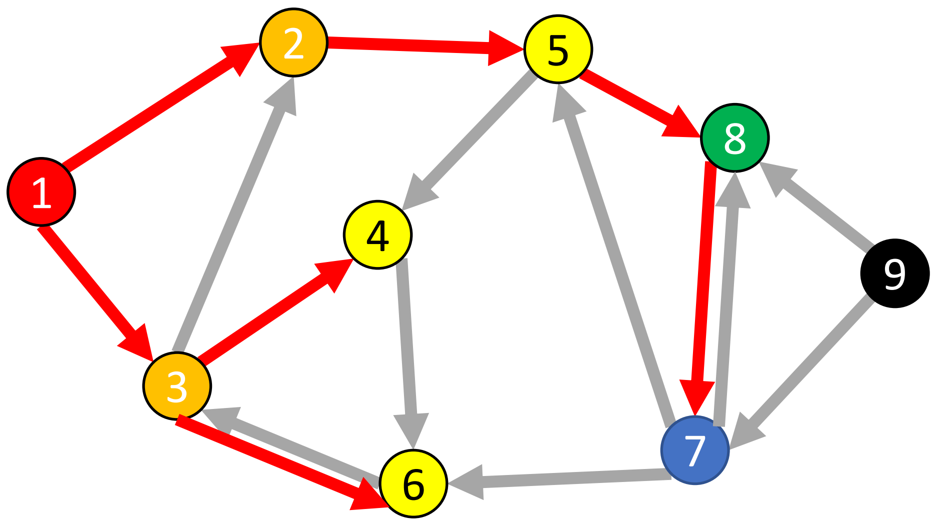
BFS



Running time: $\Theta(|V| + |E|)$

```
void bfs(graph, s){
        found = new Queue();
        found.enqueue(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.dequeue();
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                found.enqueue(v);
                        }
                }
        }
}
```

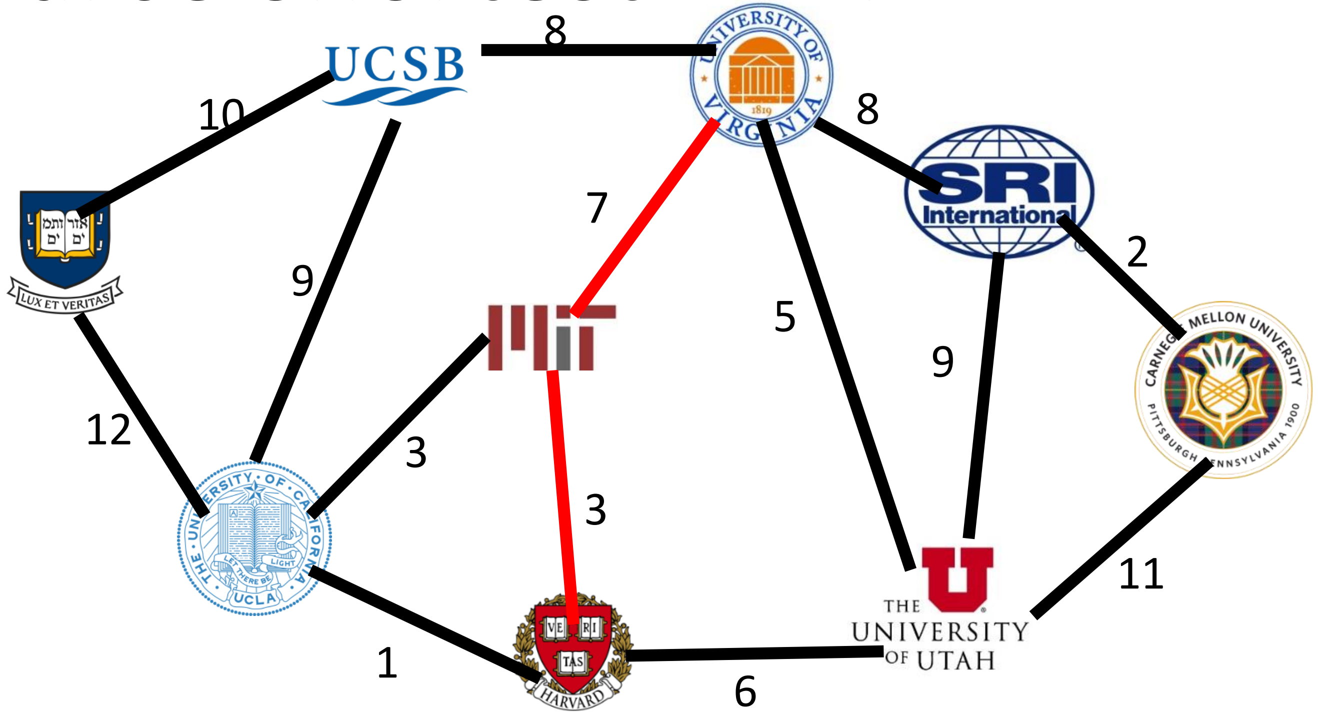
Shortest Path (unweighted)



Idea: when it's seen, remember its "layer" depth!

```
int shortestPath(graph, s, t){
        found = new Queue();
        layer = 0;
        found.enqueue(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.dequeue();
                layer = depth of current;
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                depth of v = layer + 1;
                                found.enqueue(v);
                        }
                }
        }
        return depth of t;
}
```

Single-Source Shortest Path



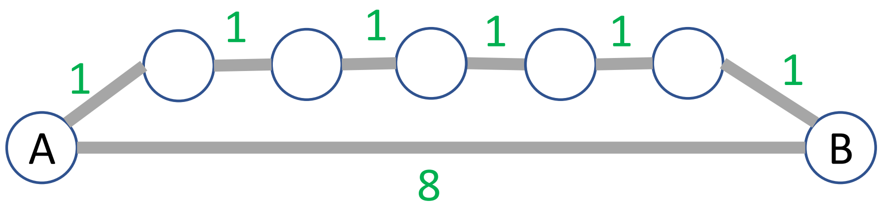
Find the quickest way to get from UVA to each of these other places

Given a graph $G = (V, E)$ and a start node $s \in V$, for each $v \in V$ find the least-weight path from $s \to v$ (call this weight $\delta(s, v)$)

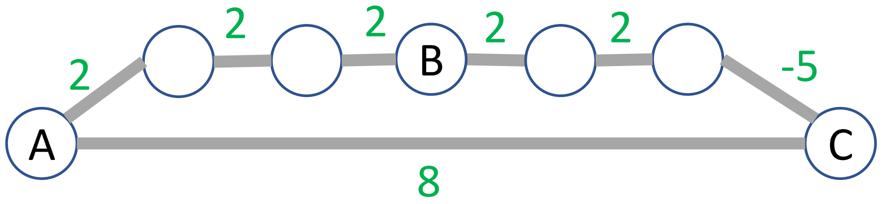
(assumption: all edge weights are positive)

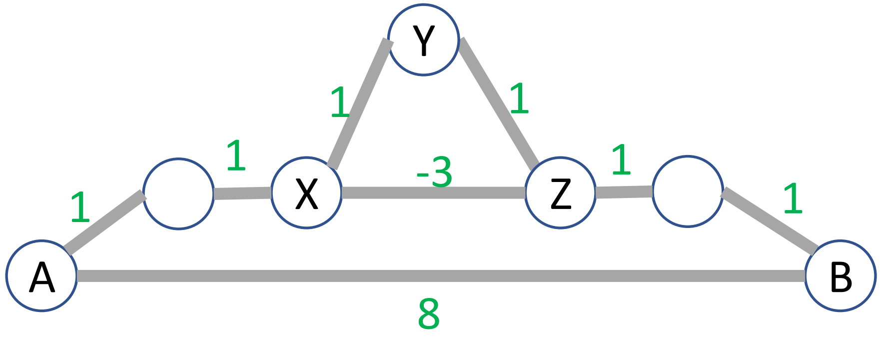
Some "Tricky" Observations

- Shortest path by sum of edge weights does not necessarily use the fewest edges.



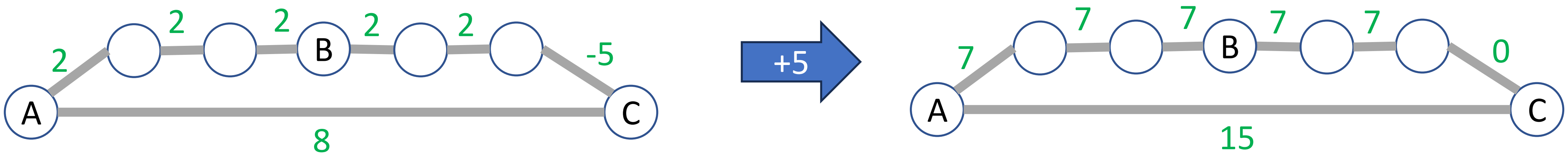
- Negative Edges:
  - Today's algorithm assumes that a path from A to B cannot be longer than a path from A to B to C.
    - Assumption is guaranteed to be true if no edges have negative weights
  - If there are negative weight cycles, problem is ill-defined





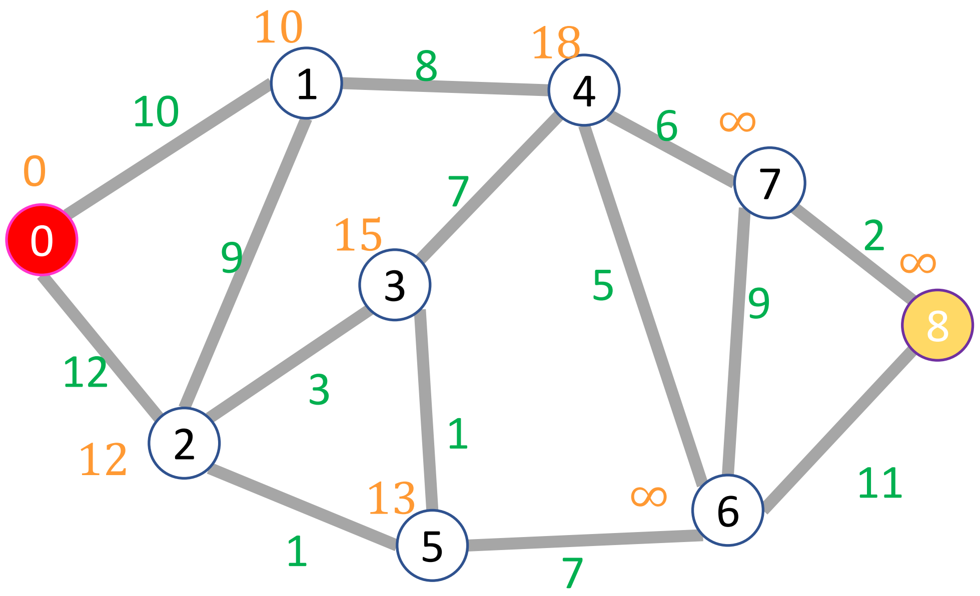
Dealing with Negative Edges (Incorrectly)

- Why doesn't this work?
  - Take the most negative edge and add it's absolute value to every other edge



Dijkstra’s Algorithm

- Input: graph with **no negative edge weights**, start node s, end node t
- Behavior: Start with node s, repeatedly go to the incomplete node “nearest” to s, stop when
- Output:
  - Distance from start to end
  - Distance from start to every node



Dijkstra's Algorithm

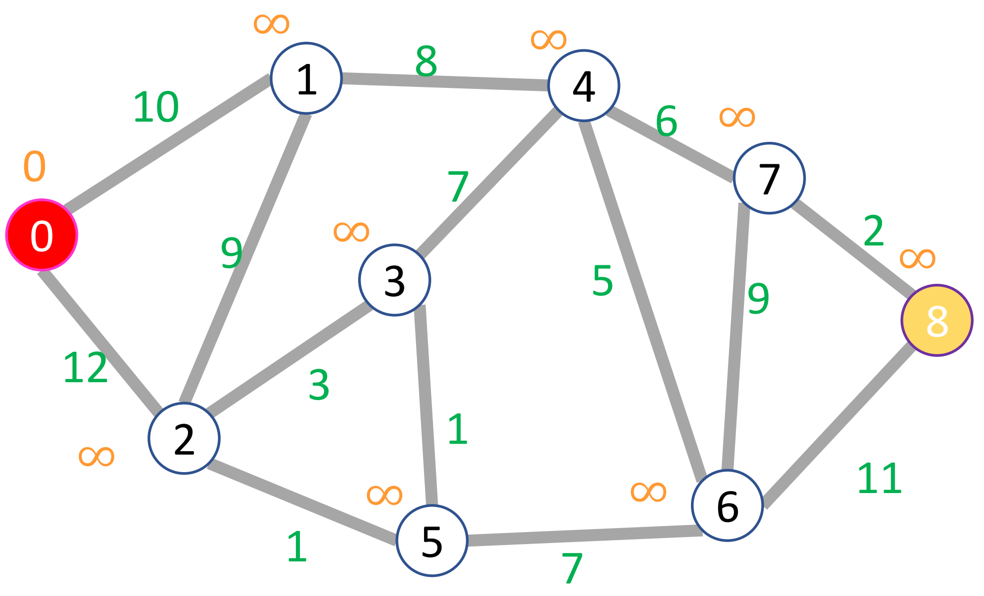
Start: 0

End: 8

Idea: When a node is the closest "unknown" node to the start, we have found its shortest path

Node	Done?
0	F
1	F
2	F
3	F
4	F
5	F
6	F
7	F
8	F

Node	Distance
0	0
1	∞
2	∞
3	∞
4	∞
5	∞
6	∞
7	∞
8	∞



Dijkstra's Algorithm

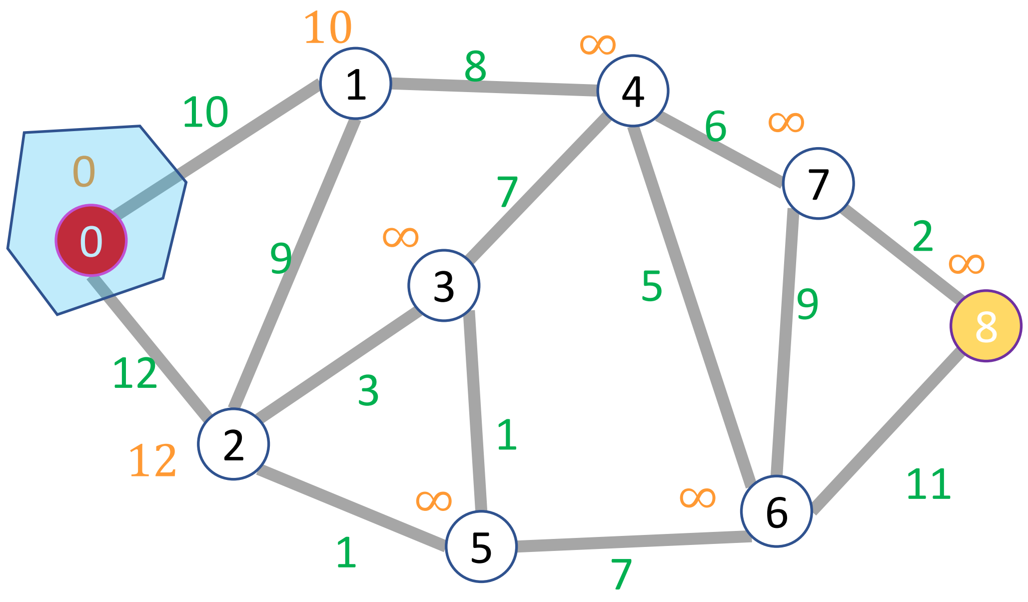
Start: 0

End: 8

Idea: When a node is the closest "unknown" node to the start, we have found its shortest path

Node	Done?
0	T
1	F
2	F
3	F
4	F
5	F
6	F
7	F
8	F

Node	Distance
0	0
1	10
2	12
3	∞
4	∞
5	∞
6	∞
7	∞
8	∞



Dijkstra's Algorithm

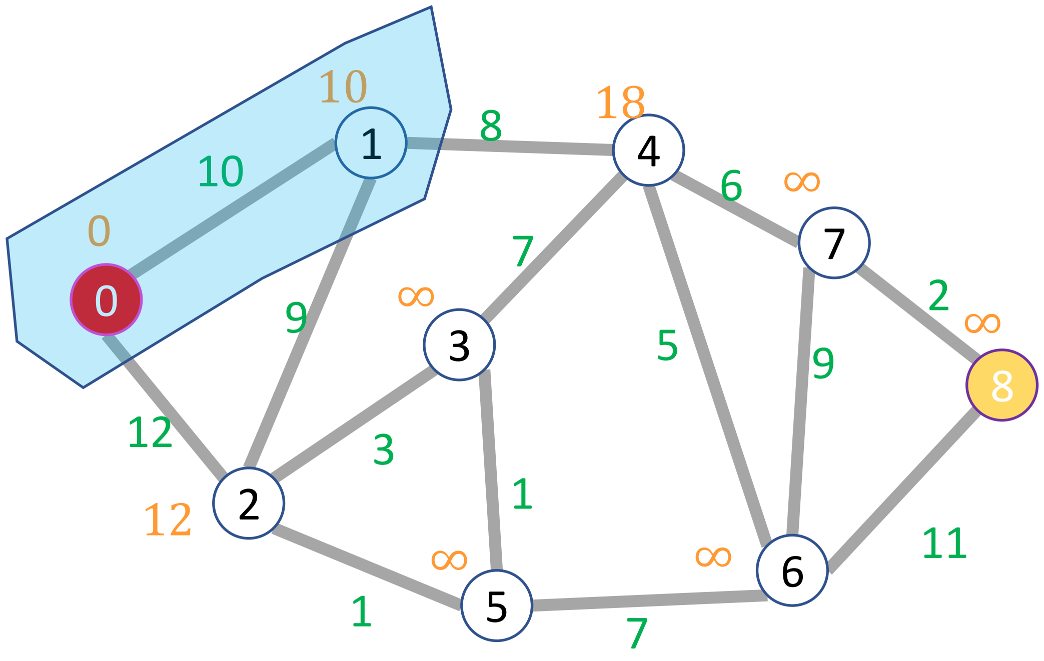
Start: 0

End: 8

Idea: When a node is the closest "unknown" node to the start, we have found its shortest path

Node	Done?
0	T
1	T
2	F
3	F
4	F
5	F
6	F
7	F
8	F

Node	Distance
0	0
1	10
2	12
3	∞
4	18
5	∞
6	∞
7	∞
8	∞



Dijkstra’s Algorithm

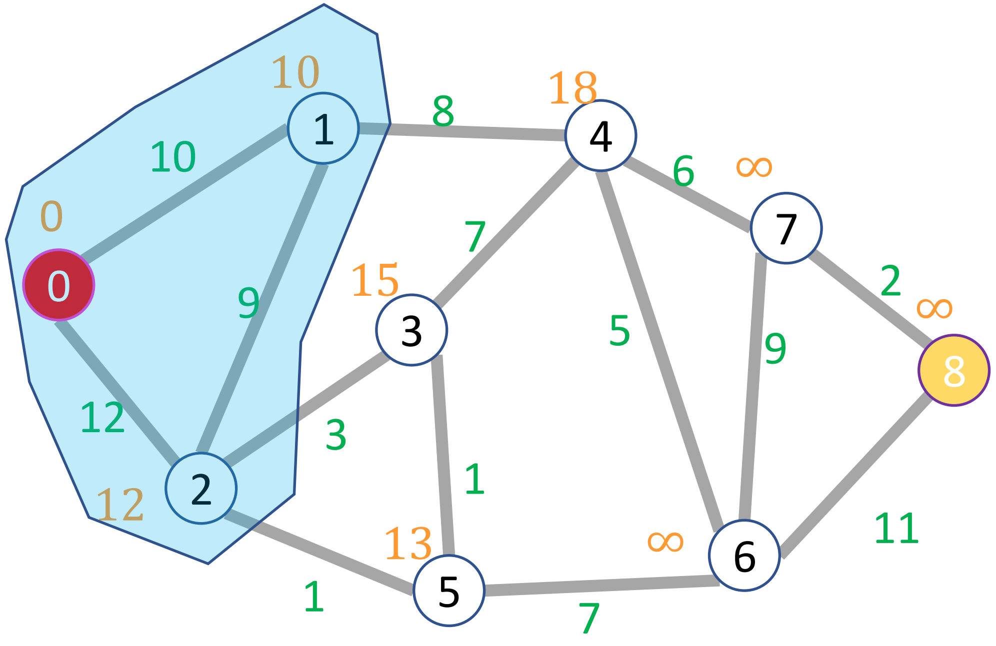
Start: 0

End: 8

Node	Done?
0	T
1	T
2	T
3	F
4	F
5	F
6	F
7	F
8	F

Node	Distance
0	0
1	10
2	12
3	15
4	18
5	13
6	∞
7	∞
8	∞

Idea: When a node is the closest “unknown” node to the start, we have found its shortest path



Dijkstra’s Algorithm

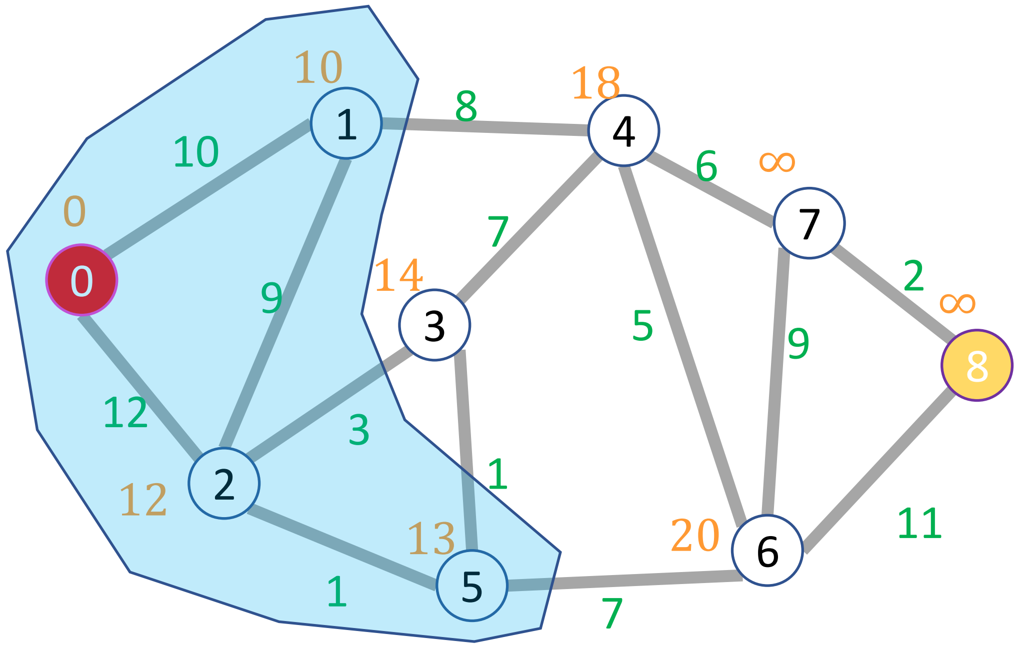
Start: 0

End: 8

Idea: When a node is the closest “unknown” node to the start, we have found its shortest path

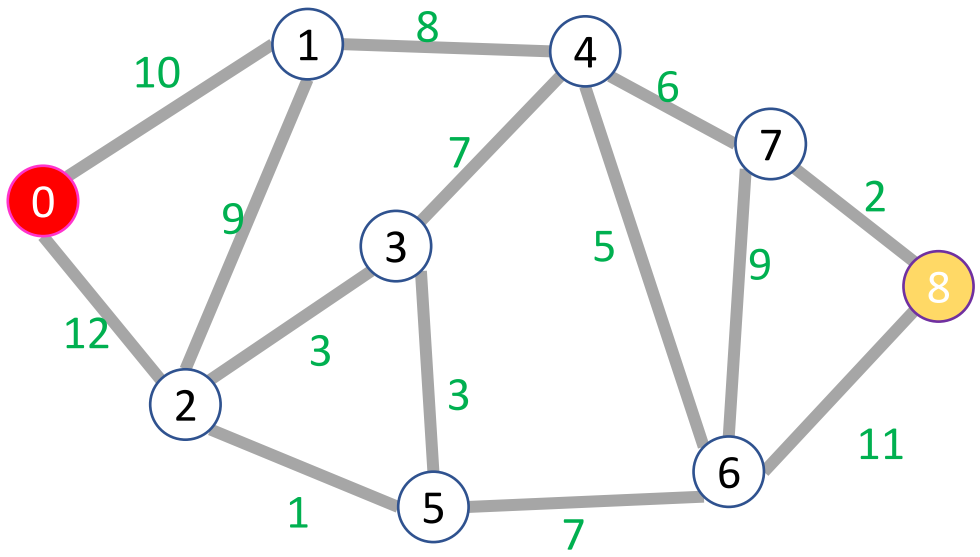
Node	Done?
0	T
1	T
2	T
3	F
4	F
5	T
6	F
7	F
8	F

Node	Distance
0	0
1	10
2	12
3	14
4	18
5	13
6	∞
7	20
8	∞



Dijkstra's Algorithm

```
int dijkstras(graph, start, end){
        PQ = new minheap();
        PQ.insert(0, start);  // priority=0, value=start
        start.distance = 0;
        while (!PQ.isEmpty){
                current = PQ.extractmin();
                if (current.known){ continue;}
                current.known = true;
                for (neighbor : current.neighbors){
                        if (!neighbor.known){
                                new_dist = current.distance + weight(current,neighbor);
                                if (new_dist < neighbor. distance){
                                        neighbor. distance = new_dist;
                                        PQ.decreaseKey(new_dist,neighbor); }
                        }
                }
        }
        return end.distance;
}
```

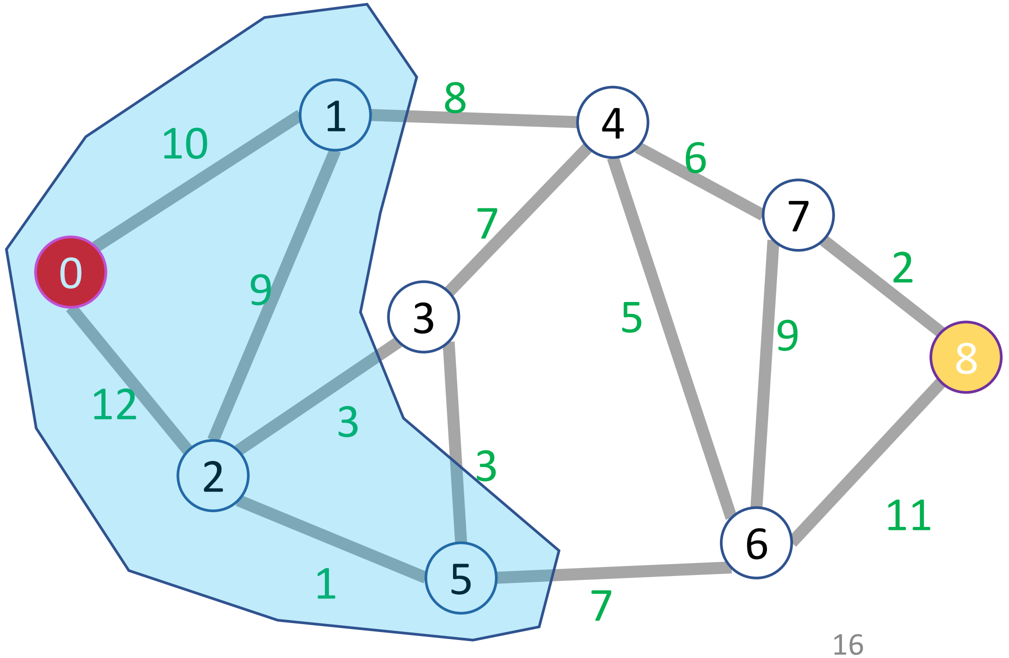


Dijkstra’s Algorithm: Running Time

- How many total priority queue operations are necessary?
  - How many times is each node added to the priority queue?
  - How many times might a node’s priority be changed?
- What’s the running time of each priority queue operation?
- Overall running time:
  - $\Theta(|E| \log|V|)$

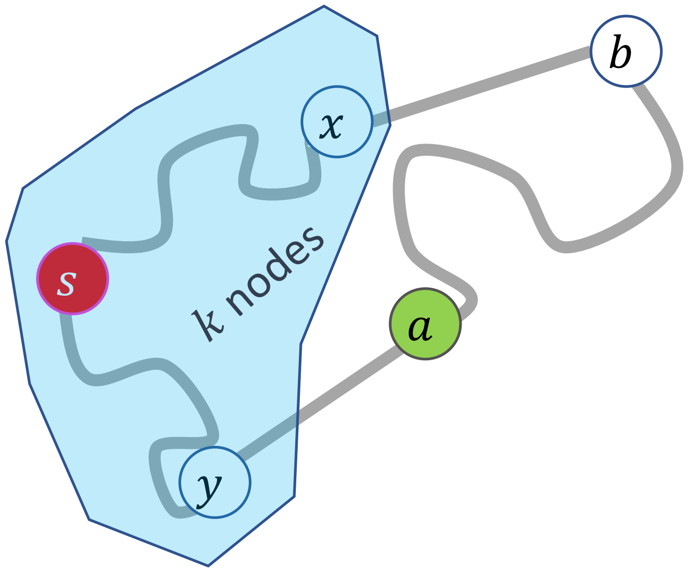
Dijkstra’s Algorithm: Correctness

- Claim: when a node is removed from the priority queue, we have found its shortest path
- Induction over number of completed nodes
- Base Case:
- Inductive Step:



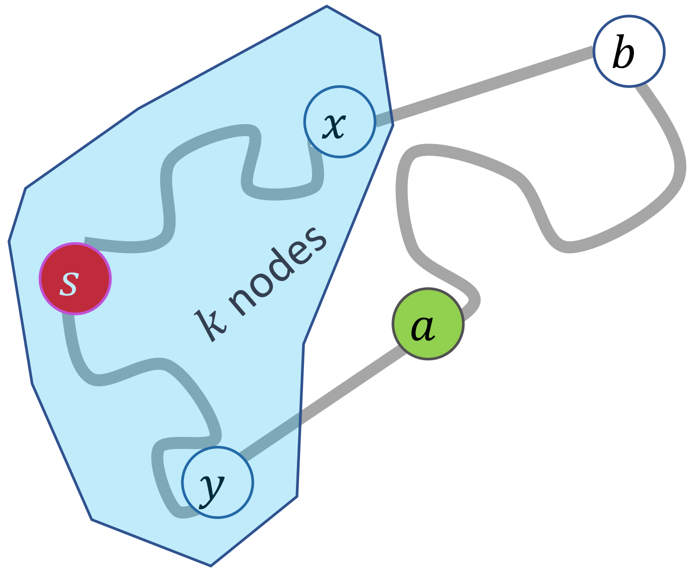
Dijkstra's Algorithm: Correctness

- Claim: when a node is removed from the priority queue, its distance is that of the shortest path
- Induction over number of completed nodes
- Base Case: Only the start node removed
  - It is indeed 0 away from itself
- Inductive Step:
  - If we have correctly found shortest paths for the first k nodes, then when we remove node $k+1$ we have found its shortest path



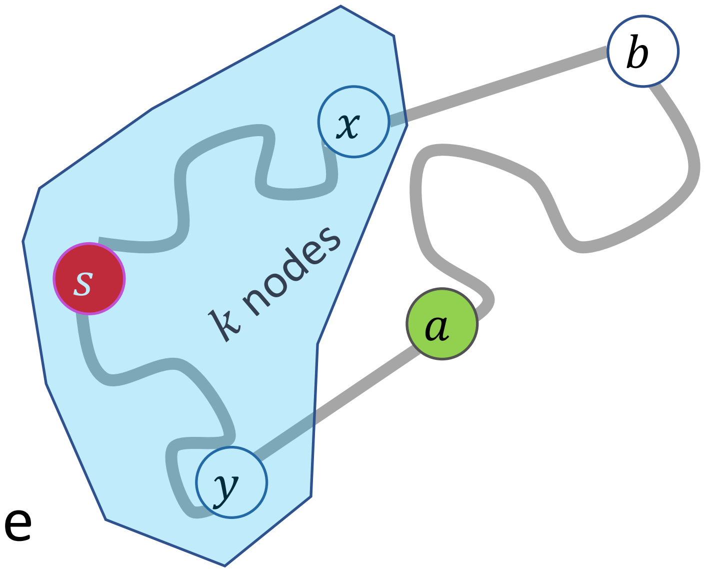
Dijkstra's Algorithm: Correctness

- Suppose a is the next node removed from the queue. What do we know bout a?



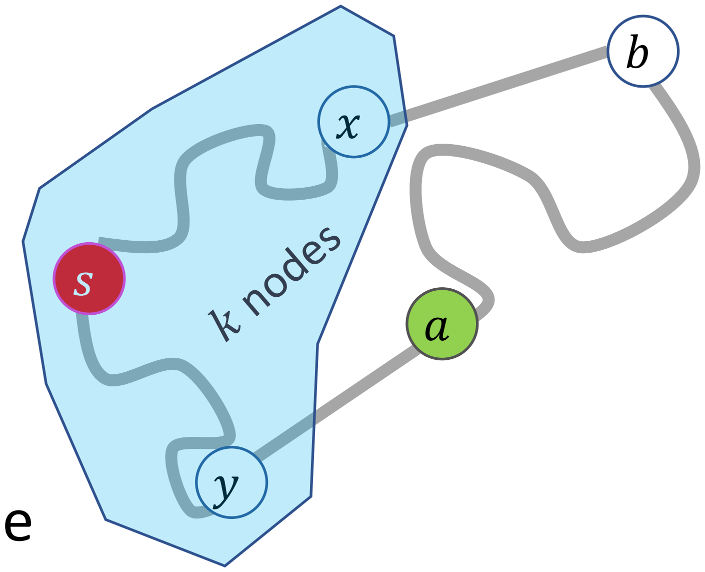
Dijkstra’s Algorithm: Correctness

- Suppose a is the next node removed from the queue.
  - No other node incomplete node has a shorter path discovered so far
- Claim: no undiscovered path to a could be shorter
  - Consider any other incomplete node b that is 1 edge away from a complete node
  - a is the closest node that is one away from a complete node
  - Thus no path that includes b can be a shorter path to a
  - Therefore the shortest path to a must use only complete nodes, and therefore we have found it already!



Dijkstra's Algorithm: Correctness

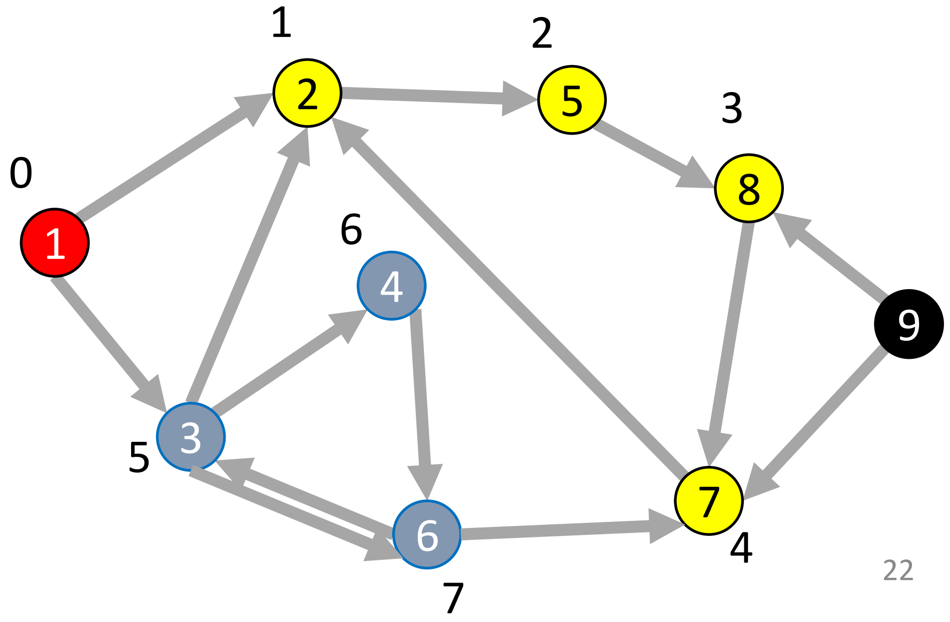
- Suppose a is the next node removed from the queue.
  - No other node incomplete node has a shorter path discovered so far
- Claim: no undiscovered path to a could be shorter
  - Consider any other incomplete node b that is 1 edge away from a complete node
  - a is the closest node that is one away from a complete node
  - No path from b to a can have negative weight
  - Thus no path that includes b can be a shorter path to a
  - Therefore the shortest path to a must use only complete nodes, and therefore we have found it already!



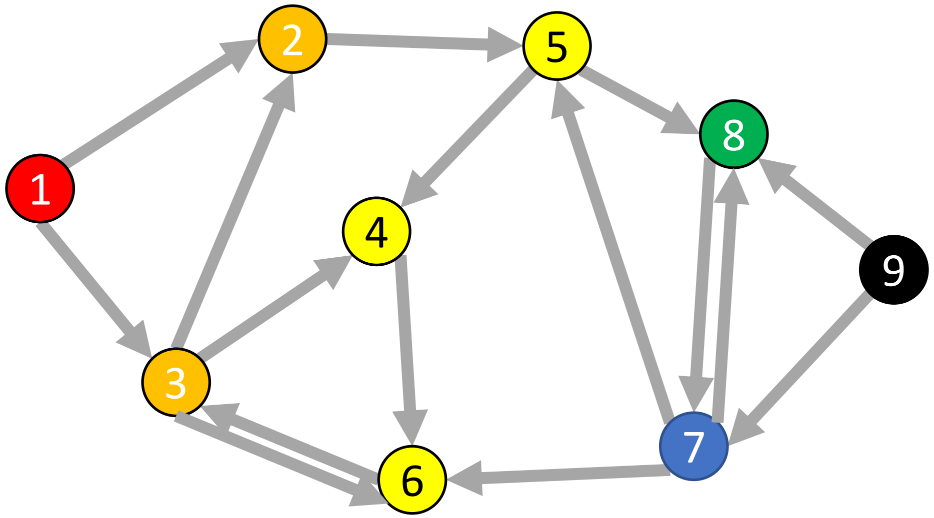
Depth-First Search

Depth-First Search

- Input: a node s
- Behavior: Start with node s, visit one neighbor of s, then all nodes reachable from that neighbor of s, then another neighbor of s,...
- Output:
  - Does the graph have a cycle?
  - A **topological sort** of the graph.



DFS (non-recursive)

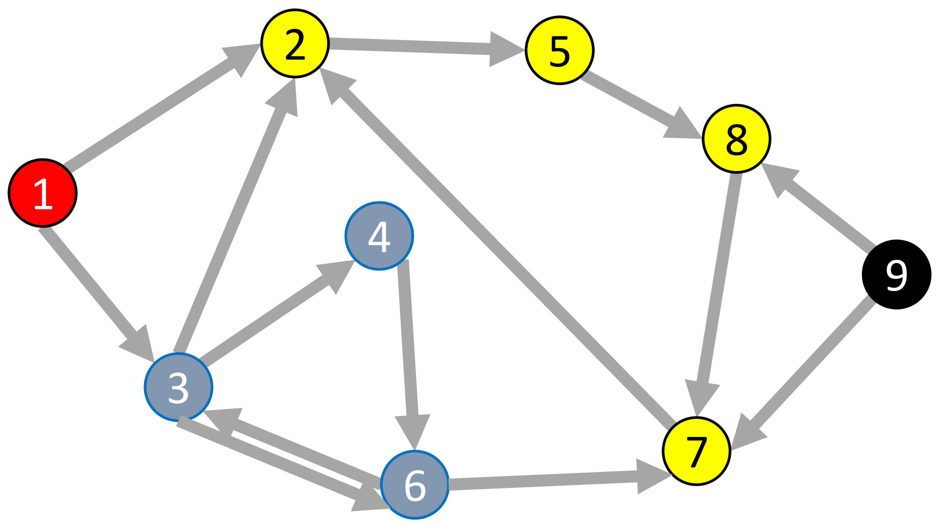


Running time: $\Theta(|V| + |E|)$

```
void dfs(graph, s){
        found = new Stack();
        found.pop(s);
        mark s as "visited";
        While (!found.isEmpty()){
                current = found.pop();
                for (v : neighbors(current)){
                        if (! v marked "visited"){
                                mark v as "visited";
                                found.push(v);
                        }
                }
        }
}
```

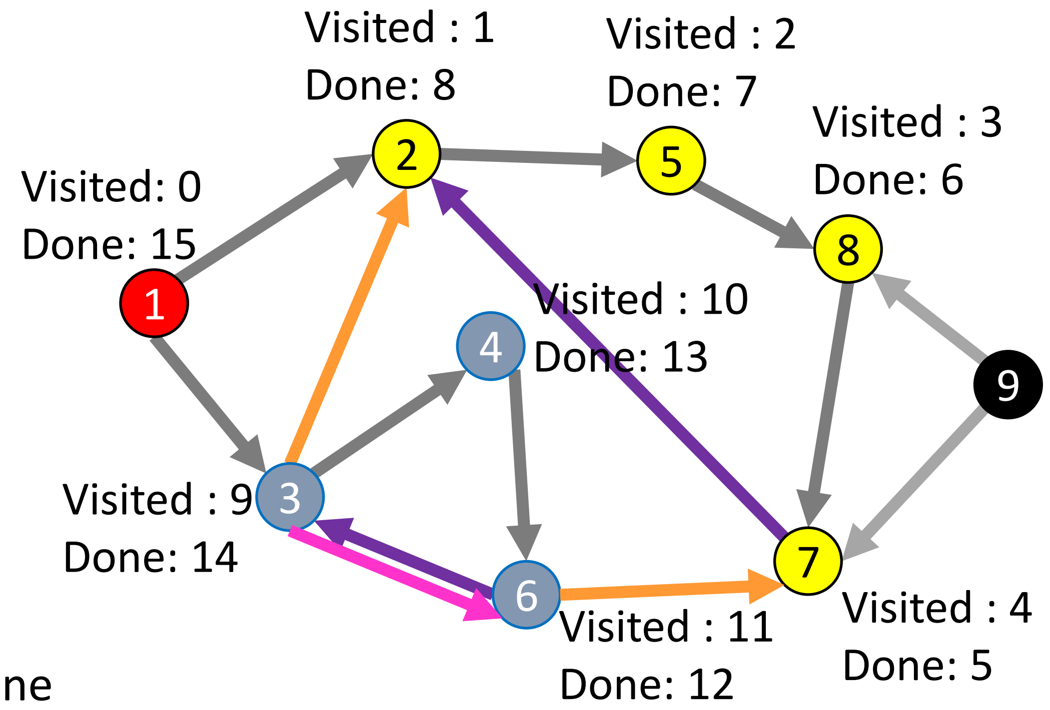
DFS Recursively (more common)

```
void dfs(graph, curr){
        mark curr as "visited";
        for (v : neighbors(current)){
                if (! v marked "visited"){
                        dfs(graph, v);
                }
        }
        mark curr as "done";
}
```



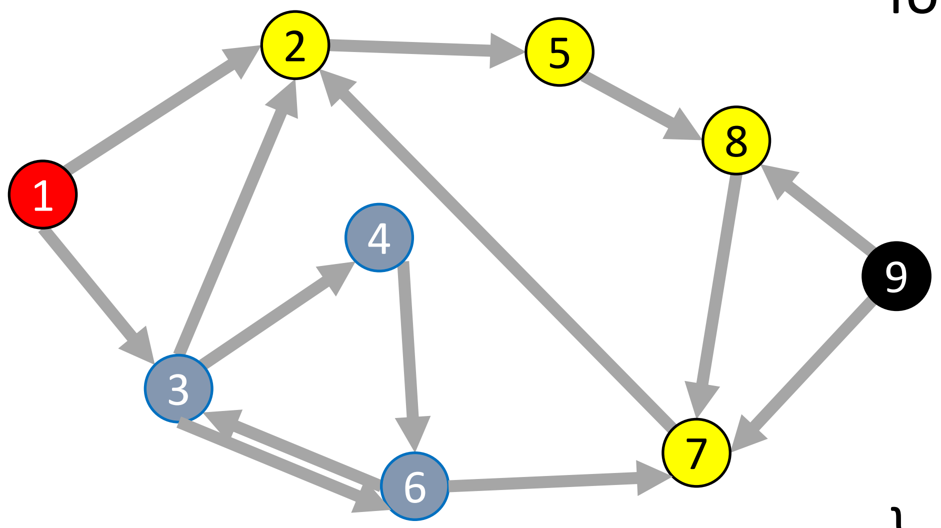
Using DFS

- Consider the "visited times" and "done times"
- Edges can be categorized:
  - Tree Edge
    - (a, b) was followed when pushing
    - (a, b) when b was unvisited when we were at a
  - Back Edge
    - (a, b) goes to an "ancestor"
    - a and b visited but not done when we saw (a, b)
    - $t_{visited}(b) < t_{visited}(a) < t_{done}(a) < t_{done}(b)$
  - Forward Edge
    - (a, b) goes to a "descendent"
    - b was visited and done between when a was visited and done
    - $t_{visited}(a) < t_{visited}(b) < t_{done}(b) < t_{done}(a)$
  - Cross Edge
    - (a, b) goes to a node that doesn't connect to a
    - b was seen and done before a was ever visited
    - $t_{done}(b) < t_{visited}(a)$



Cycle Detection

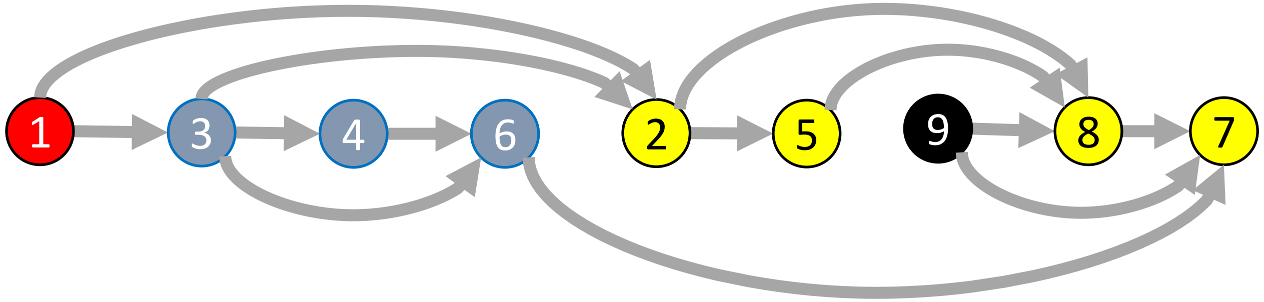
Idea: Look for a back edge!

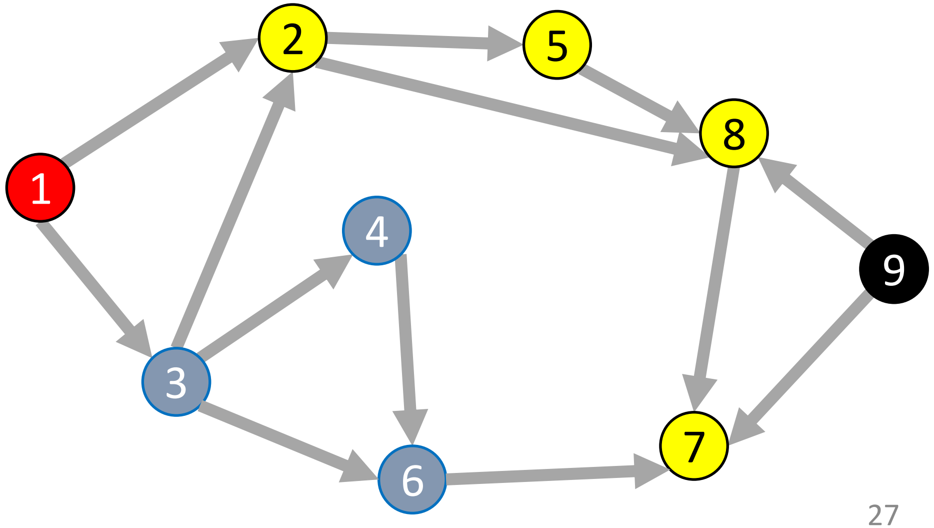



```
boolean hasCycle(graph, curr){
        mark curr as "visited";
        cycleFound = false;
        for (v : neighbors(current)){
                if (v marked "visited" && ! v marked "done"){
                        cycleFound=true;
                }
                if (! v marked "visited" && !cycleFound){
                        cycleFound = hasCycle(graph, v);
                }
        }
        mark curr as "done";
        return cycleFound;
}
```

Topological Sort

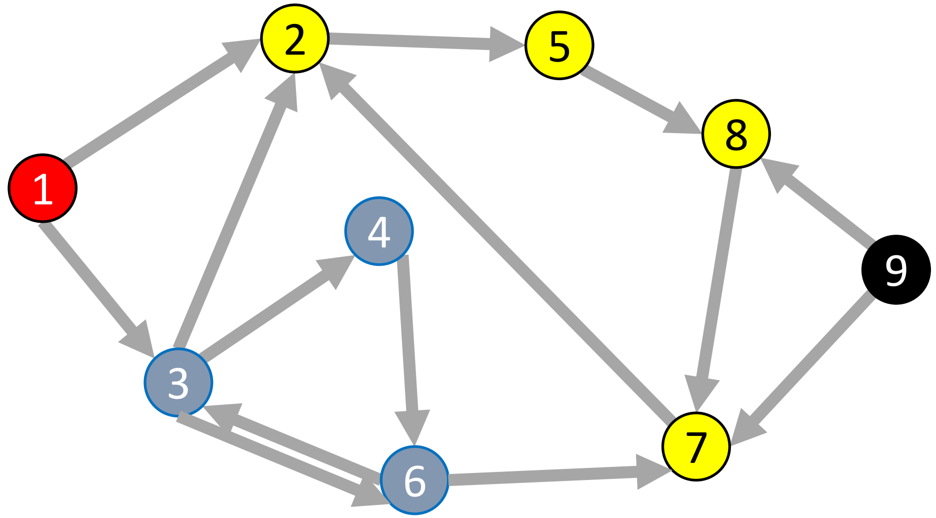
- A Topological Sort of a **directed acyclic graph** $\boldsymbol{G = (V, E)}$ is a permutation of V such that if $(u, v) \in E$ then u is before v in the permutation





DFS Recursively

```
void dfs(graph, curr){
        mark curr as "visited";
        for (v : neighbors(current)){
                if (! v marked "visited"){
                        dfs(graph, v);
                }
        }
        mark curr as "done";
}
```

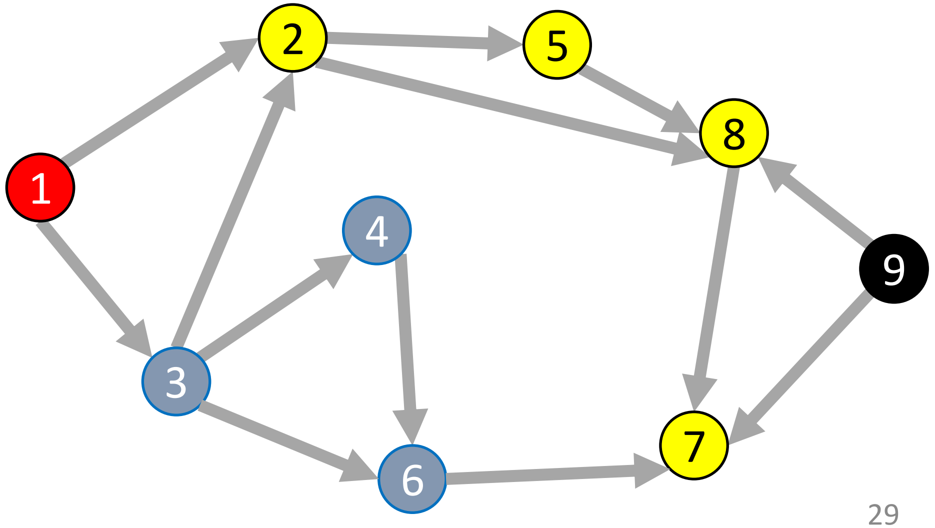


DFS: Topological sort

```
def dfs(graph, s):
        seen = [False, False, False, …] # length matches |V|
        done = [False, False, False, …] # length matches |V|
        dfs_rec(graph, s, seen, done)

def dfs_rec(graph, curr, seen, done):
        mark curr as seen
        for v in neighbors(current):
                if v not seen:
                        dfs_rec(graph, v, seen, done)
        mark curr as done
```

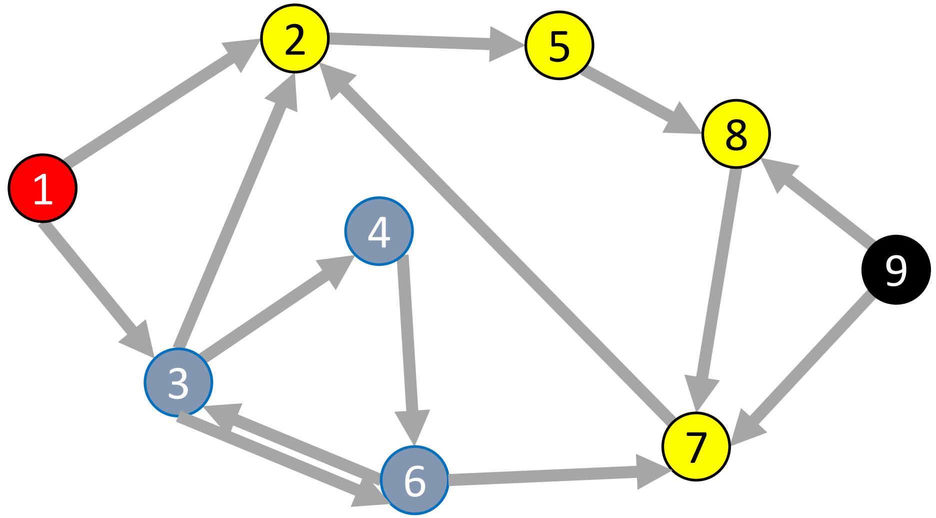
Idea: List in reverse order by finish time



DFS Recursively

```
void dfs(graph, curr){
        mark curr as "visited";
        for (v : neighbors(current)){
                if (! v marked "visited"){
                        dfs(graph, v);
                }
        }
        mark curr as "done";
}
```

Idea: List in reverse order by finish time



DFS: Topological sort

```
List topSort(graph){
        List<Nodes> finished = new List<>();
        for (Node v : graph.vertices){
                if (!v.visited){
                        finishTime(graph, v, finished);
                }
        }
        finished.reverse();
        return finished;
}

void finishTime(graph, curr, finished){
        curr.visited = true;
        for (Node v : curr.neighbors){
                if (!v.visited){
                        finishTime(graph, v, finished);
                }
        }
        finished.add(curr)
}
```

Idea: List in reverse order by finish time

finished: