CSE 332: Data Structures & Parallelism

Lecture 12: Comparison Sorting

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Today

- Sorting
  - Comparison sorting

Introduction to sorting

- Stacks, queues, priority queues, and dictionaries all focused on providing one element at a time
- But often we know we want “all the data items” in some order
  - Anyone can sort, but a computer can sort faster
  - Very common to need data sorted somehow
    - Alphabetical list of people
    - Population list of countries
    - Search engine results by relevance
    - …
- Different algorithms have different asymptotic and constant-factor trade-offs
  - No single ‘best’ sort for all scenarios
  - Knowing one way to sort just isn’t enough

More reasons to sort

General technique in computing:

Preprocess (e.g. sort) data to make subsequent operations faster

Example: Sort the data so that you can

- Find the **`k`**th largest in constant time for any **`k`**
- Perform binary search to find an element in logarithmic time

Whether the benefit of the preprocessing depends on

- How often the data will change
- How much data there is

The main problem, stated carefully

For now we will assume we have *n* comparable elements in an array and we want to rearrange them to be in increasing order

Input:

- An array **`A`** of data records
- A key value in each data record
- A comparison function (consistent and total)
  - Given keys a & b, what is their relative ordering? <, =, >?
  - Ex: keys that implement Comparable or have a Comparator that can handle them

Effect:

- Reorganize the elements of **`A`** such that for any **`i`** and **`j`**,

  if **`i < j`** then **`A[i]`** ≤ **`A[j]`**
- Usually unspoken assumption: **`A`** must have all the same data it started with
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort

Variations on the basic problem

1. Maybe elements are in a linked list (could convert to array and back in linear time, but some algorithms needn't do so)
2. Maybe in the case of ties we should preserve the original ordering
   – Sorts that do this naturally are called stable sorts
   – One way to sort twice, Ex: Sort movies by year, then for ties, alphabetically
3. Maybe we must not use more than $O(1)$ "auxiliary space"
   – Sorts meeting this requirement are called 'in-place' sorts
   – Not allowed to allocate extra array (at least not with size O(n)), but can allocate O(1) # of variables
   – All work done by swapping around in the array
4. Maybe we can do more with elements than just compare
   – Comparison sorts assume we work using a binary 'compare' operator
   – In special cases we can sometimes get faster algorithms
5. Maybe we have too much data to fit in memory
   – Use an "external sorting" algorithm

Sorting: The Big Picture

Simple algorithms: O(n^2)

Insertion sort
Selection sort
Shell sort
…

Fancier algorithms: O(n log n)

Heap sort
Merge sort
Quick sort (avg)
…

Comparison lower bound: Ω(n log n)

Specialized algorithms: O(n)

Bucket sort
Radix sort

Handling huge data sets

External sorting

Insertion Sort

- Idea: At step **`k`**, put the **`k`**th element in the correct position among the first **`k`** elements

- Alternate way of saying this:
  - Sort first two elements
  - Now insert 3rd element in order
  - Now insert 4th element in order
  - …

- "Loop invariant": when loop index is **`i`**, first **`i`** elements are sorted

- Time?

  Best-case ______ Worst-case ______ "Average" case _____

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- Time?

  Best-case O(n) Worst-case $O(n^2)$ "Average" case $O(n^2)$

  start sorted start reverse sorted (see text)

Selection sort

- Idea: At step **`k`**, find the smallest element among the not-yet-sorted elements and put it at position k
- Alternate way of saying this:
  - Find smallest element, put it 1st
  - Find next smallest element, put it 2nd
  - Find next smallest element, put it 3rd
  - …
- "Loop invariant": when loop index is **`i`**, first **`i`** elements are the **`i`** smallest elements in sorted order
- Time?

  Best-case ______ Worst-case ______ "Average" case _____

Selection sort

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- "Loop invariant": when loop index is **`i`**, first **`i`** elements are the **`i`** smallest elements in sorted order

- Time?

  Best-case $O(n^2)$ Worst-case $O(n^2)$ "Average" case $O(n^2)$

  Always $T(1) = 1$ and $T(n) = n + T(n-1)$

Insertion Sort vs. Selection Sort

- Different algorithms
- Solve the same problem
- Have the same worst-case and average-case asymptotic complexity
  - Insertion-sort has better best-case complexity; preferable when input is “mostly sorted”
- Other algorithms are more efficient *for non-small arrays that are not already almost sorted*
  - Insertion sort may do well on small arrays

Aside: We won't cover Bubble Sort

- It doesn't have good asymptotic complexity: $O(n^2)$
- It's not particularly efficient with respect to common factors
- Basically, almost everything it is good at, some other algorithm is at least as good at
- Some people seem to teach it just because someone taught it to them

- For fun see: "Bubble Sort: An Archaeological Algorithmic Analysis", Owen Astrachan, SIGCSE 2003 http://www.cs.duke.edu/~ola/bubble/bubble.pdf

Sorting: The Big Picture

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Fancier algorithms: O(n log n)

Heap sort
Merge sort
Quick sort (avg)
…

Comparison lower bound: Ω(n log n)

Specialized algorithms: O(n)

Bucket sort
Radix sort

Handling huge data sets

External sorting

Heap sort

- Sorting with a heap is easy:
  - **`insert`** each **`arr[i]`**, better yet use **`buildHeap`**
  - **`for(i=0; i < arr.length; i++)`**

```
    arr[i] = deleteMin();
```

- Worst-case running time:

- We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There’s a trick to make it in-place…

Heap sort

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  - `for(i=0; i < arr.length; i++)`

```
      arr[i] = deleteMin();
```

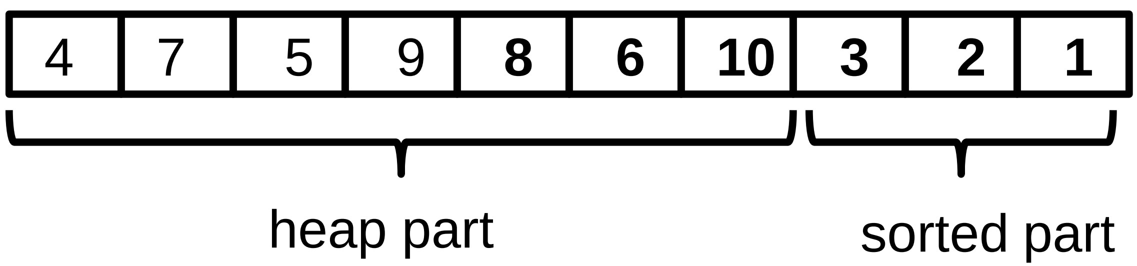
- Worst-case running time: $O(n$ **`log`** $n)$ why?

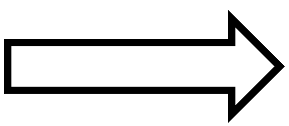
- We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There’s a trick to make it in-place…

In-place heap sort

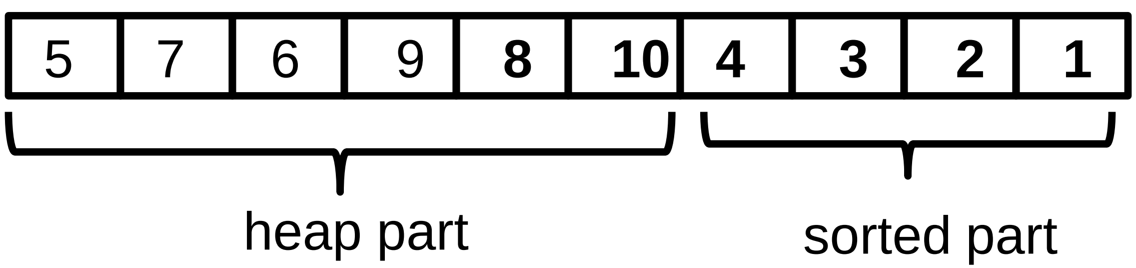
But this reverse sorts – how would you fix that?

- Treat the initial array as a heap (via **`buildHeap`**)
- When you delete the **`i`**th element, put it at **`arr[n-i]`**
  - It's not part of the heap anymore!





`arr[n-i]=`
`deleteMin()`



“AVL sort”

- How?

“AVL sort”

- We can also use a balanced tree to:
  - `insert` each element: total time $O(n \log n)$
  - Do an in-order traversal $O(n)$

- But this cannot be made in-place and has worse constant factors than heap sort
  - both are $O(n \log n)$ in worst, best, and average case
  - neither parallelizes well
  - heap sort is better

- Don’t even think about trying to sort with a hash table…

Divide and conquer

Very important technique in algorithm design

1. Divide problem into smaller parts

2. Solve the parts independently
   - Think recursion
   - Or potential parallelism

3. Combine solution of parts to produce overall solution

Ex: Sort each half of the array, combine together; to sort each half, split into halves…

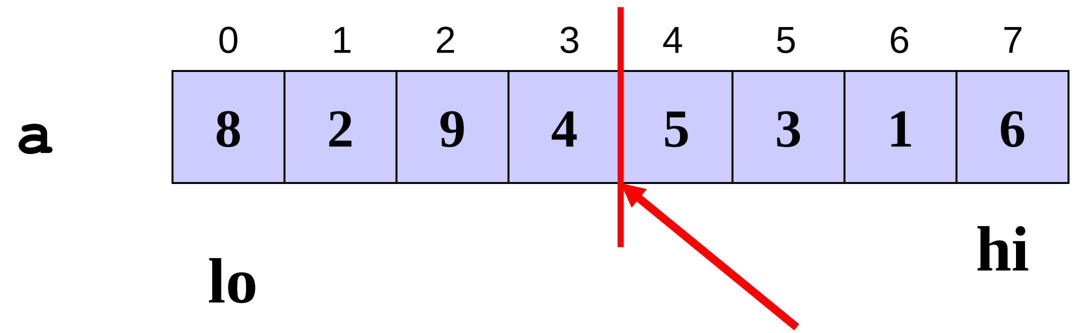
Divide-and-conquer sorting

Two great sorting methods are fundamentally divide-and-conquer

1. Mergesort: Sort the left half of the elements (recursively)
   Sort the right half of the elements (recursively)
   Merge the two sorted halves into a sorted whole

2. Quicksort: Pick a “pivot” element
   Divide elements into those less-than pivot
   and those greater-than pivot
   Sort the two divisions (recursively on each)
   Answer is [*sorted-less-than* then *pivot* then
   sorted-greater-than]

Mergesort



- To sort array from position **`lo`** to position **`hi`**:
  - If range is 1 element long, it's sorted! (Base case)
  - Else, split into two halves:
    - Sort from **`lo`** to **`(hi+lo)/2`**
    - Sort from **`(hi+lo)/2`** to **`hi`**
    - Merge the two halves together

- Merging takes two sorted parts and sorts everything
  - $O(n)$ but requires auxiliary space…

Example, focus on merging

Start with:

a

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After we return from left and right recursive calls (pretend it works for now)

a

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 "fingers"

and 1 more array

aux

(After merge, copy back to original array)

Example, focus on merging

Start with:

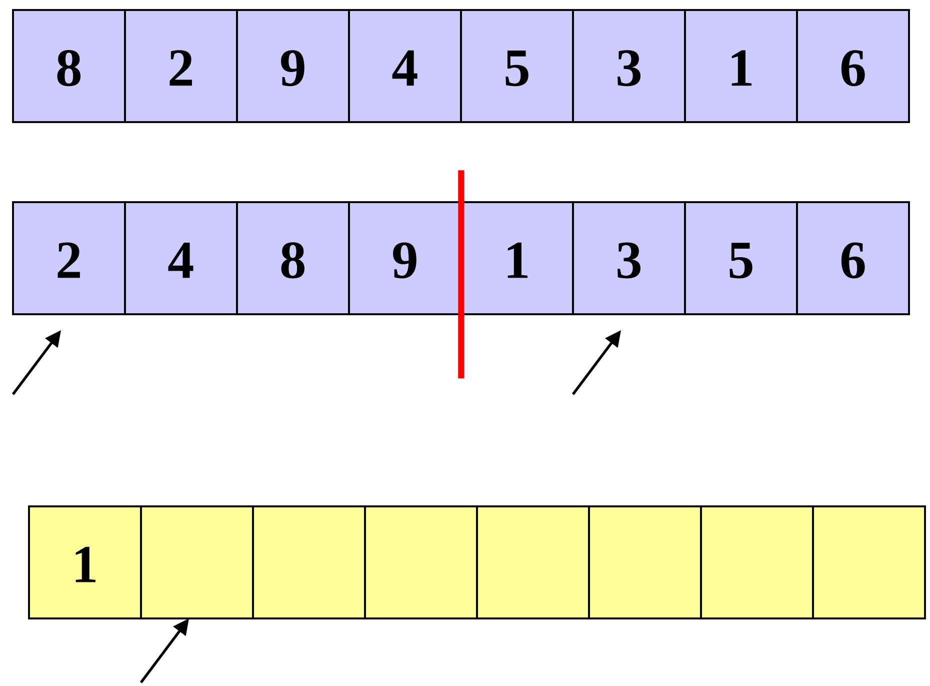
After recursion:

(not magic ☺)

Merge:

Use 3 “fingers”

and 1 more array



(After merge,
copy back to
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Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:

(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 "fingers"

and 1 more array

1	**2**						

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:

(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 “fingers”

and 1 more array

1	**2**	**3**					

(After merge,
copy back to
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Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:

(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 “fingers”

and 1 more array

1	**2**	**3**	**4**				

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:

(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 “fingers”

and 1 more array

1	**2**	**3**	**4**	**5**			

(After merge, copy back to original array)

Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:
(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:
Use 3 “fingers”
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1	**2**	**3**	**4**	**5**	**6**		

(After merge,
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Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:
(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:
Use 3 “fingers”
and 1 more array

1	**2**	**3**	**4**	**5**	**6**	**8**	

(After merge,
copy back to
original array)

Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:

(not magic ☺)

2	**4**	**8**	**9**	**1**	**3**	**5**	**6**

Merge:

Use 3 “fingers”

and 1 more array

1	**2**	**3**	**4**	**5**	**6**	**8**	**9**

(After merge,
copy back to
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Example, focus on merging

Start with:

8	**2**	**9**	**4**	**5**	**3**	**1**	**6**

After recursion:
(not magic ☺)

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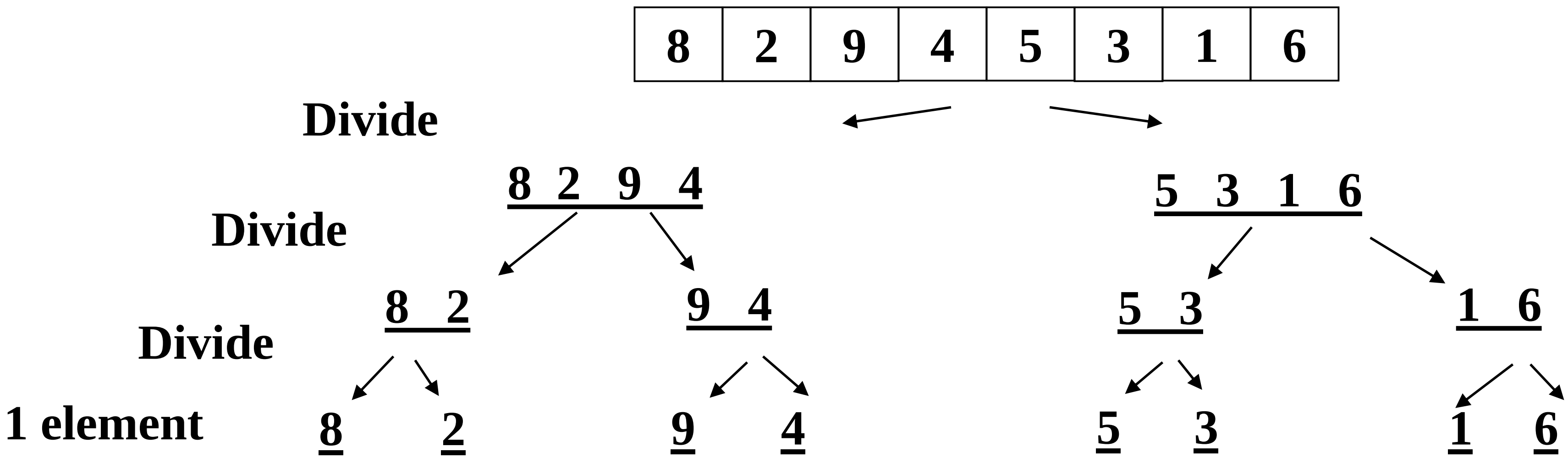
Merge:
Use 3 “fingers”
and 1 more array

1	**2**	**3**	**4**	**5**	**6**	**8**	**9**

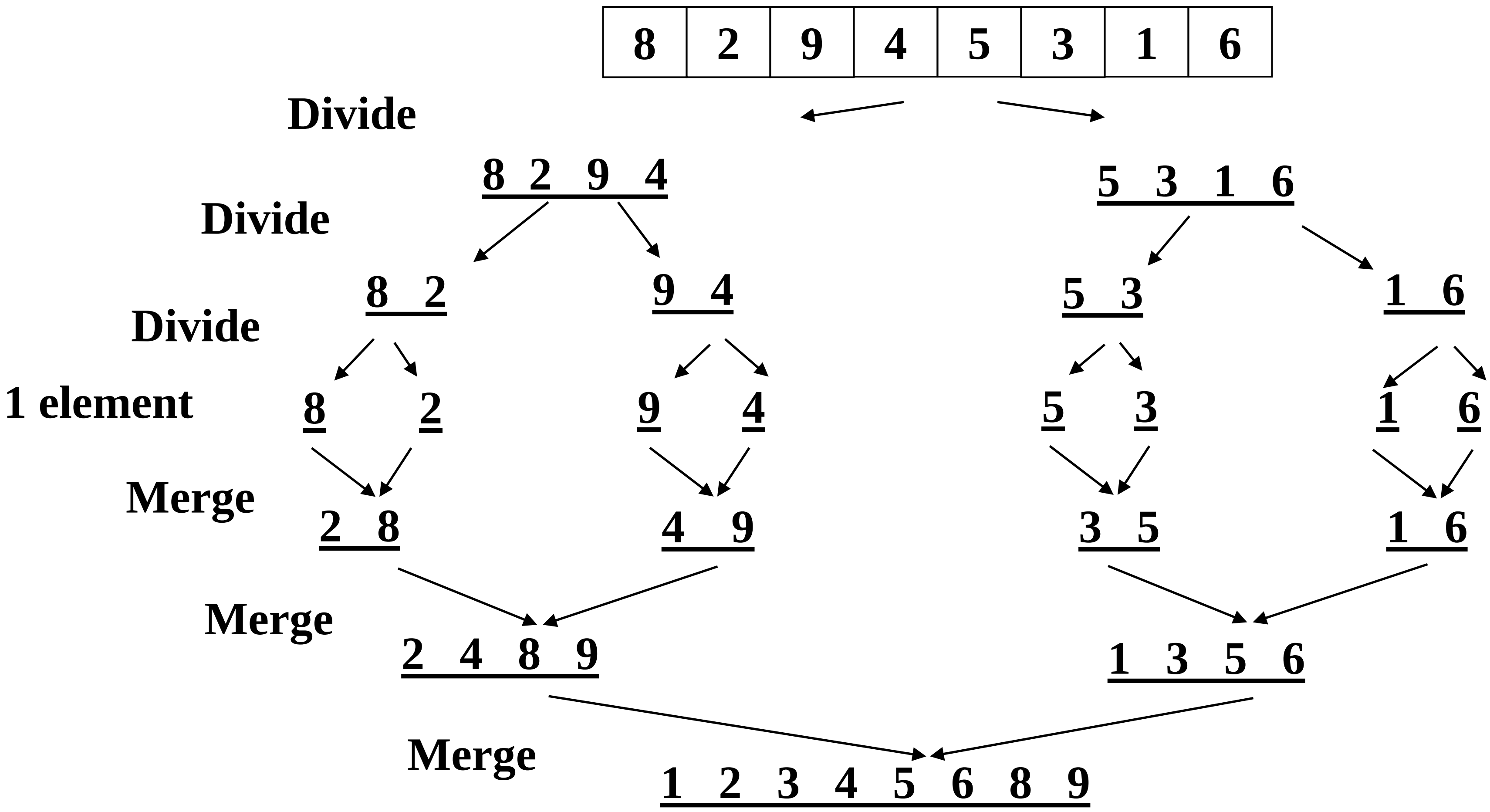
(After merge,
copy back to
original array)

1	**2**	**3**	**4**	**5**	**6**	**8**	**9**

Mergesort example: Recursively splitting list in half

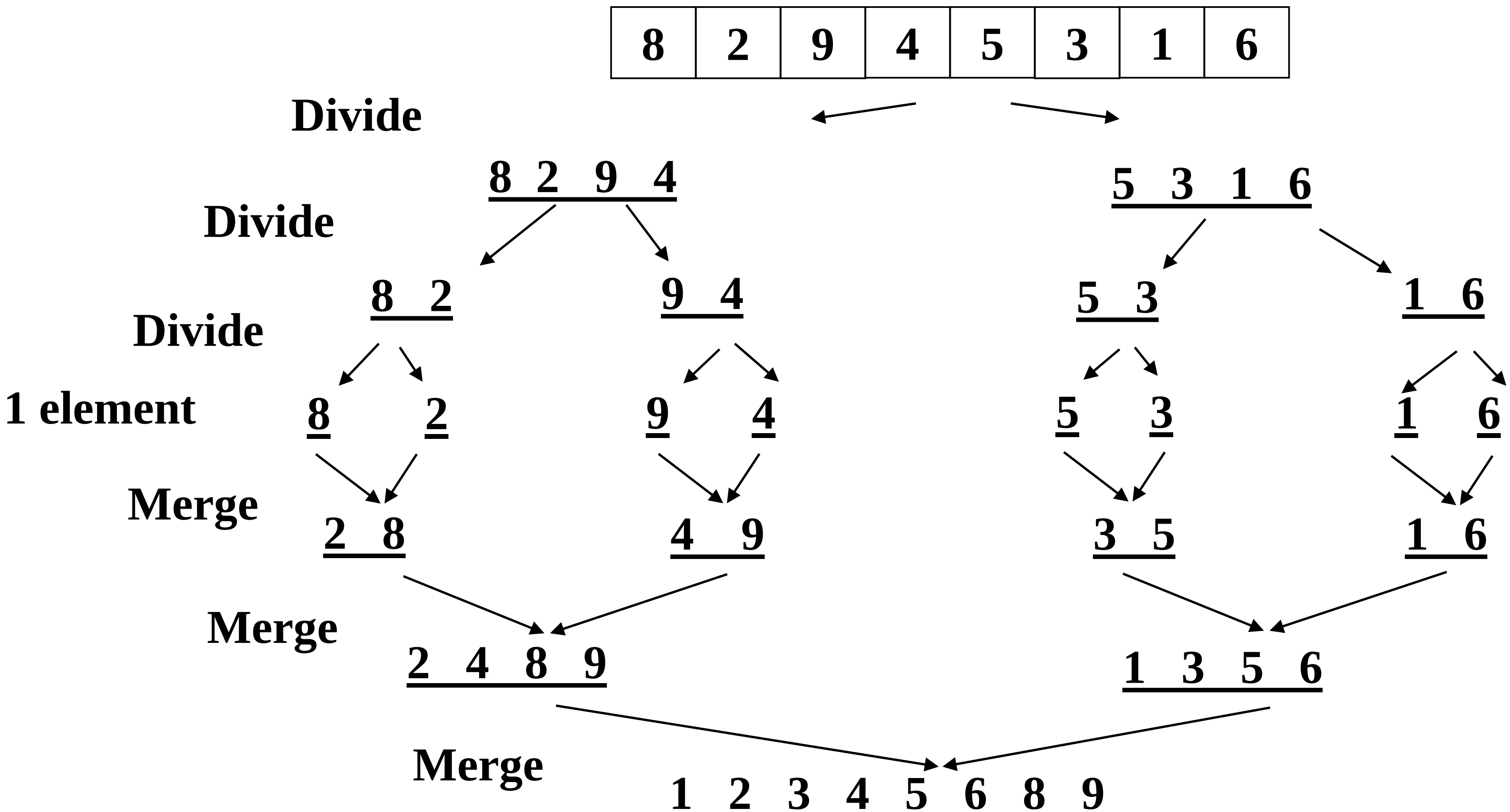


Mergesort example: Merge as we return from recursive calls



When a recursive call ends, it's sub-arrays are each in order; just need to merge them in order together

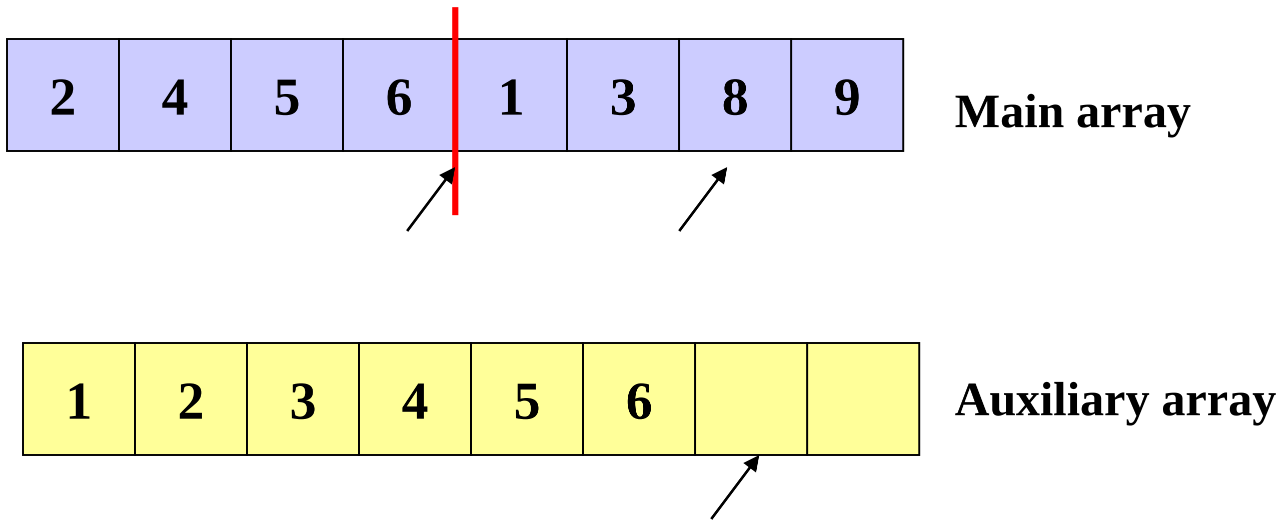
Mergesort example: Merge as we return from recursive calls



We need another array in which to do each merging step; merge results into there, then copy back to original array

Mergesort, some details: saving a little time

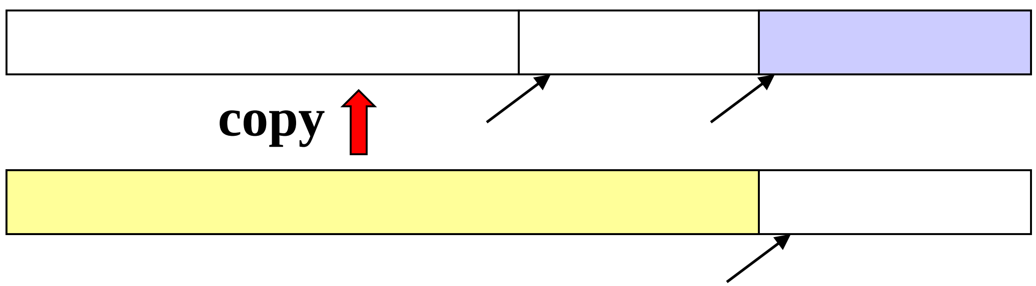
- What if the final steps of our merging looked like the following:



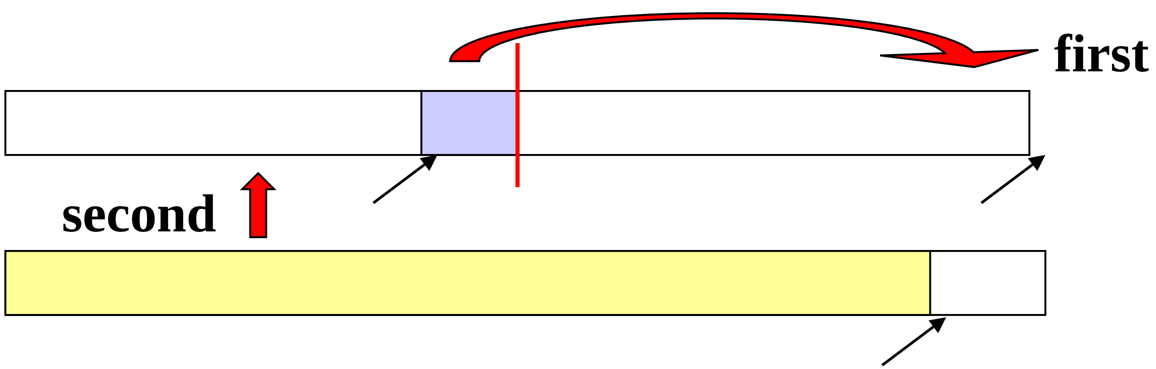
- Seems kind of wasteful to copy 8 & 9 to the auxiliary array just to copy them immediately back…

Mergesort, some details: saving a little time

- Unnecessary to copy 'dregs' over to auxiliary array
  - If left-side finishes first, just stop the merge & copy the auxiliary array:



  - If right-side finishes first, copy dregs directly into right side, then copy auxiliary array



Some details: saving space / copying

Simplest / worst approach:

Use a new auxiliary array of size **`(hi-lo)`** for every merge

Returning from a recursive call? Allocate a new array!

Better:

Reuse same auxiliary array of size **`n`** for every merging stage

Allocate auxiliary array at beginning, use throughout

Best (but a little tricky):

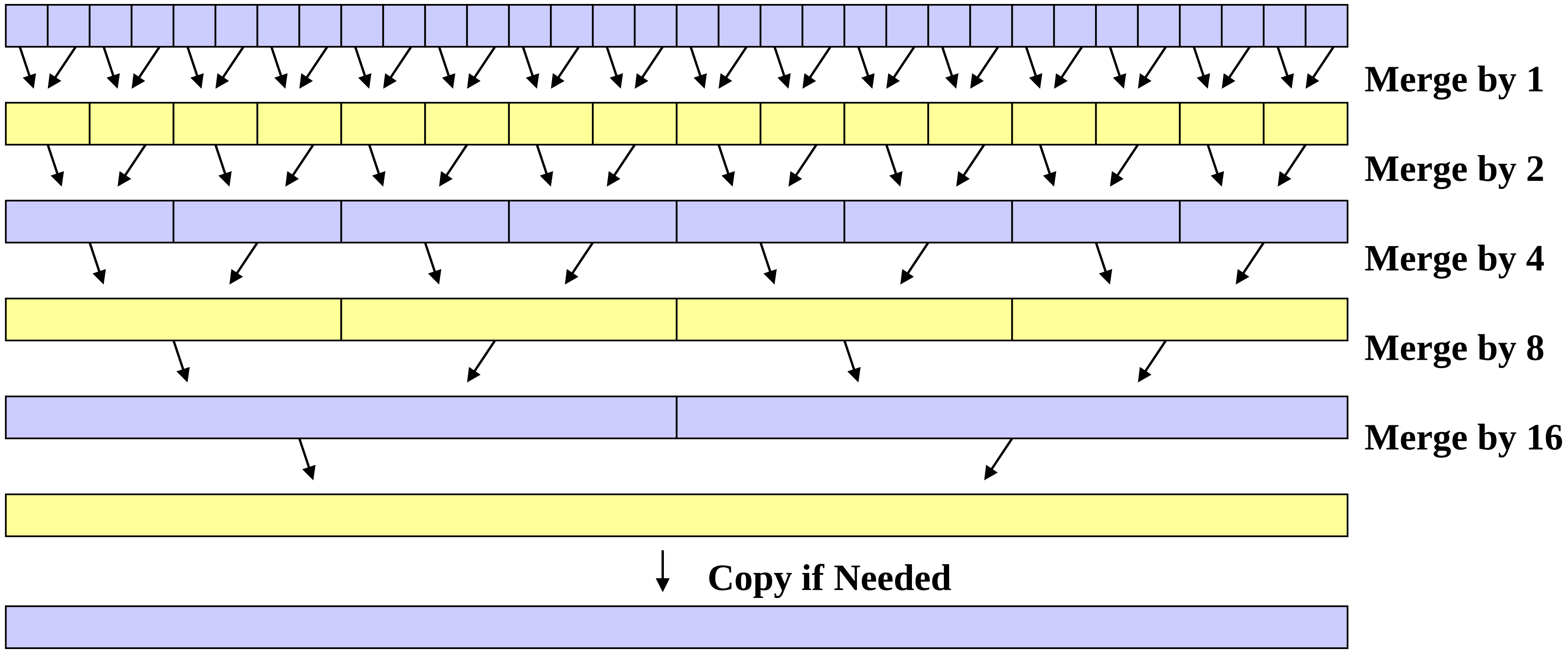
Don’t copy back – at 2nd, 4th, 6th, … merging stages, use the original array as the auxiliary array and vice-versa

- Need one copy at end if number of stages is odd

Picture of the "best" from previous slide: Allocate one auxiliary array, switch each step

First recurse down to lists of size 1

As we return from the recursion, switch off arrays



Arguably easier to code up without recursion at all

Linked lists and big data

We defined the sorting problem as over an array, but sometimes you want to sort linked lists

One approach:

- Convert to array: $O(n)$
- Sort: $O(n \, \mathtt{log} \, n)$
- Convert back to list: $O(n)$

Or: mergesort works very nicely on linked lists directly

- heapsort and quicksort do not
- insertion sort and selection sort do but they're slower

Mergesort is also the sort of choice for external sorting

- Linear merges minimize disk accesses

Mergesort Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time (and space):

To sort n elements, we:

- Return immediately if n=1
- Else do 2 subproblems of size n/2 and then an $O(n)$ merge

Recurrence relation?

Mergesort Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time (and space):

To sort n elements, we:

- Return immediately if $n=1$
- Else do 2 subproblems of size $n/2$ and then an $O(n)$ merge

Recurrence relation:

$$T(1) = c_1$$

$$T(n) = 2T(n/2) + c_2 n$$

MergeSort Recurrence

(For simplicity let constants be 1 – no effect on asymptotic answer)

$T(1) = 1$

$T(n) = 2T(n/2) + n$

$= 2(2T(n/4) + n/2) + n$

$= 4T(n/4) + 2n$

$= 4(2T(n/8) + n/4) + 2n$

$= 8T(n/8) + 3n$

…. (after k expansions)

$= 2^{k}T(n/2^{k}) + kn$

So total is $2^{k}T(n/2^{k}) + kn$ where

$n/2^{k} = 1$, i.e., $\log n = k$

That is, $2^{\log n}\, T(1) + n \log n$

$= n + n \log n$

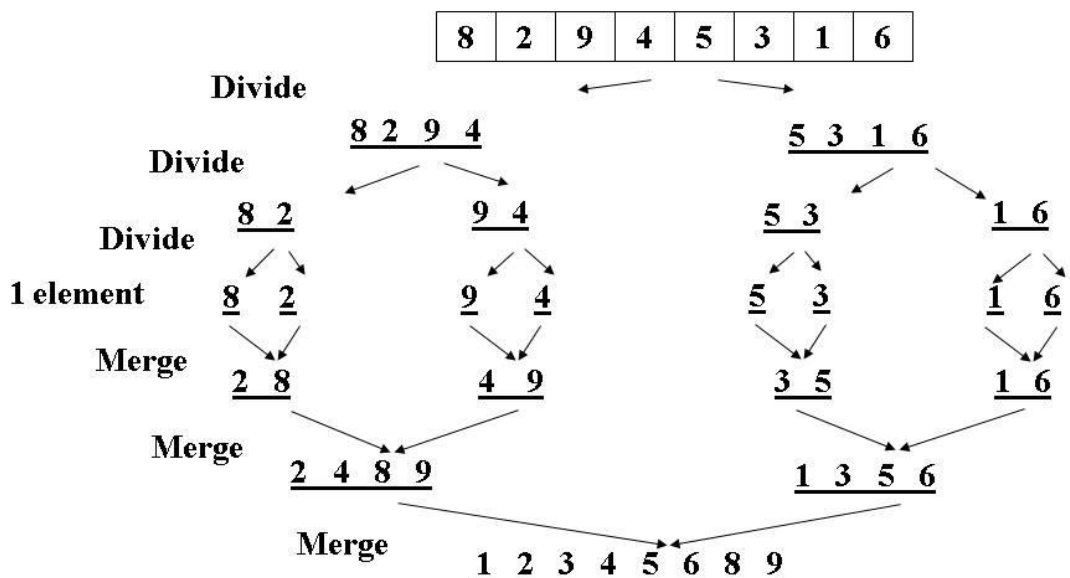
$= O(n \log n)$

Or more intuitively…

This recurrence comes up often enough you should just “know” it’s *O*(*n* **`log`** *n*)

Merge sort is relatively easy to intuit (best, worst, and average):

- The recursion “tree” will have **`log`** *n* height
- At each level we do a *total* amount of merging equal to *n*



Quicksort

- Also uses divide-and-conquer
  - Recursively chop into halves
  - But, instead of doing all the work as we merge together, we'll do all the work as we recursively split into halves
  - Also unlike MergeSort, does not need auxiliary space

- $O(n \texttt{ log } n)$ on average ☺, but $O(n^2)$ worst-case ☹
  - MergeSort is always O(nlogn)
  - So why use QuickSort?

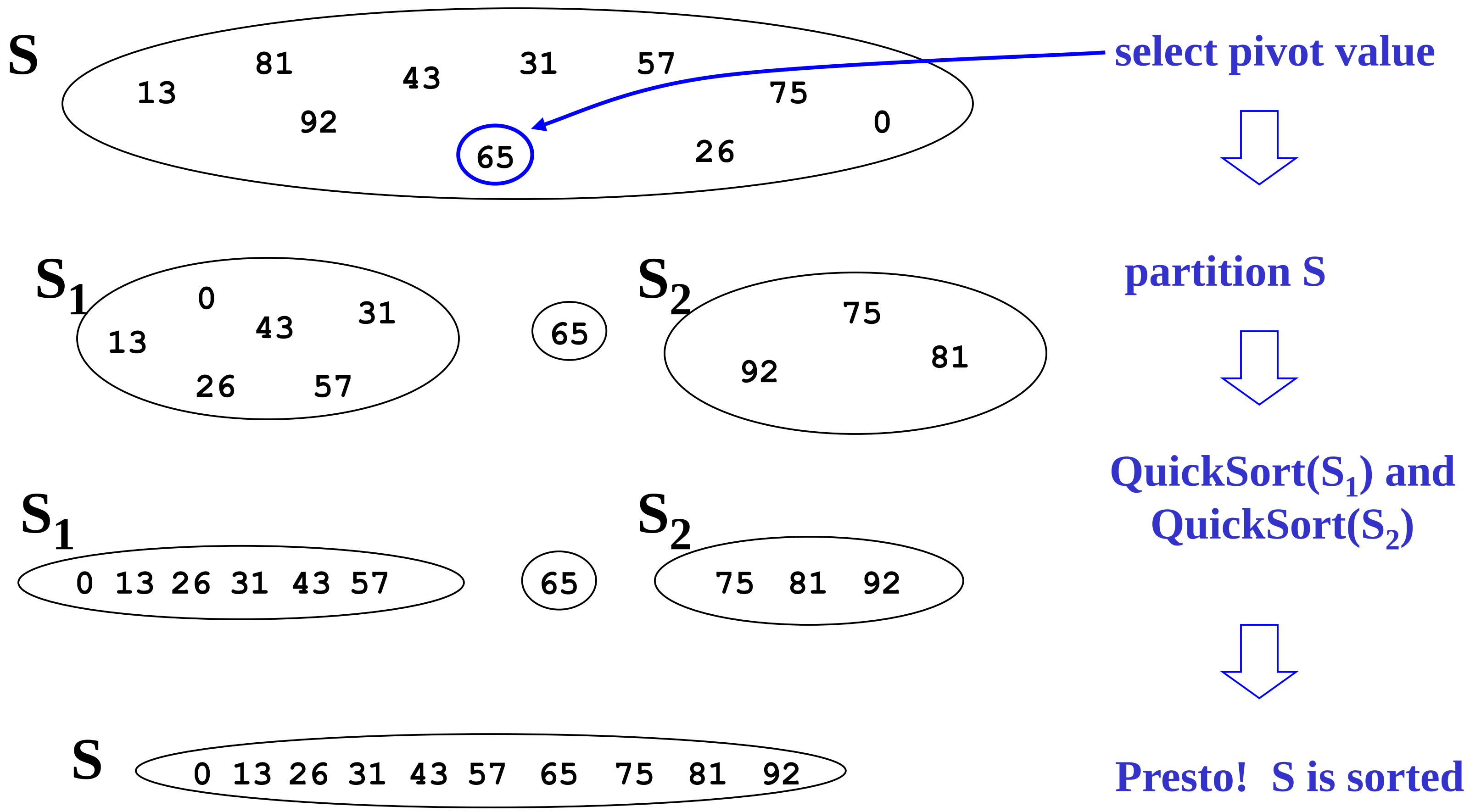
- Can be faster than mergesort
  - Often believed to be faster
  - Quicksort does fewer copies and more comparisons, so it depends on the relative cost of these two operations!

Quicksort Overview

1. Pick a pivot element
   - Hopefully an element ~median
   - Good QuickSort performance depends on good choice of pivot; we'll see why later, and talk about good pivot selection later
2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot
3. Recursively sort A and C
4. The answer is, "as simple as A, B, C"

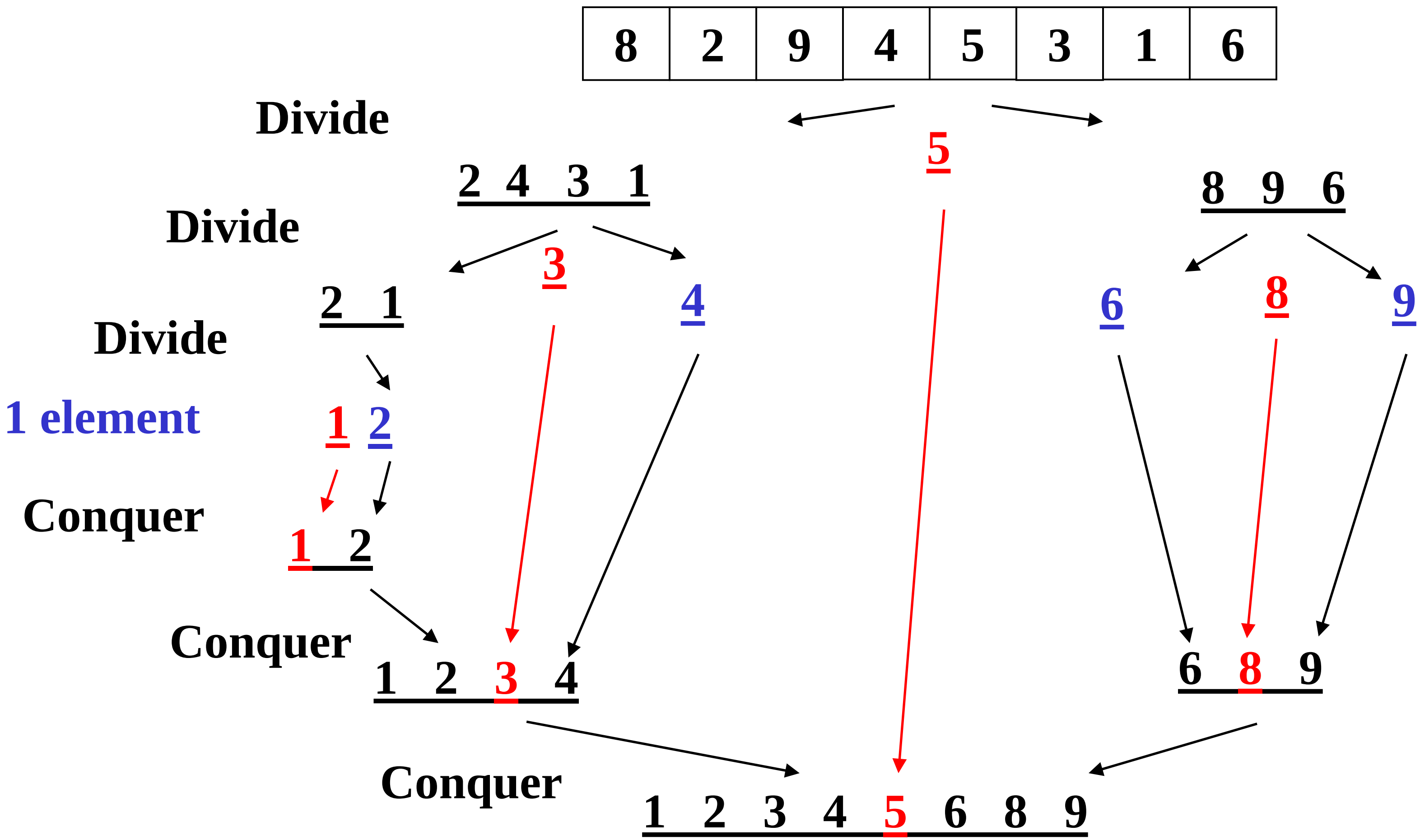
(Alas, there are some details lurking in this algorithm)

Quicksort: Think in terms of sets

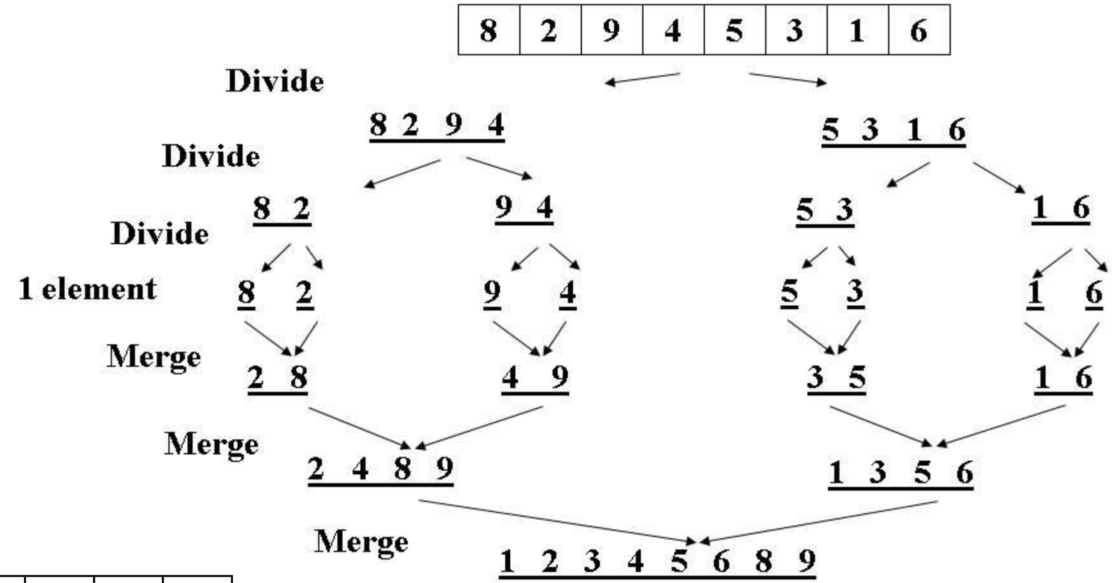


[Weiss]

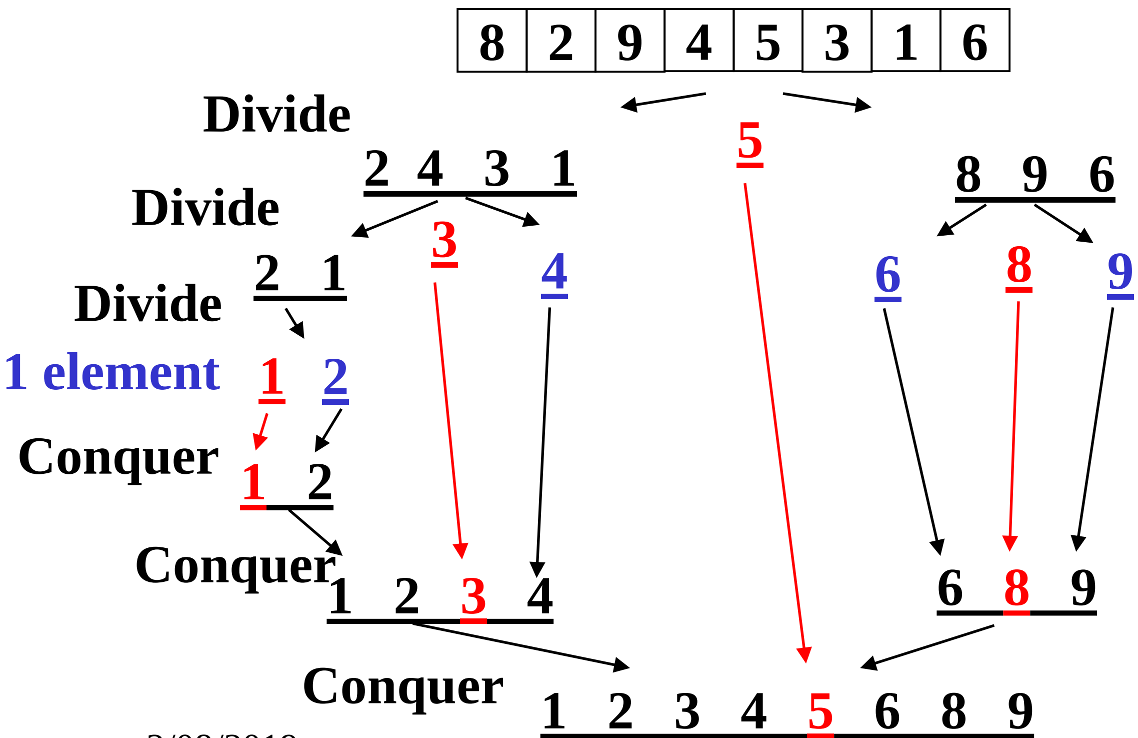
Quicksort Example, showing recursion



MergeSort Recursion Tree



QuickSort Recursion Tree



Quicksort Details

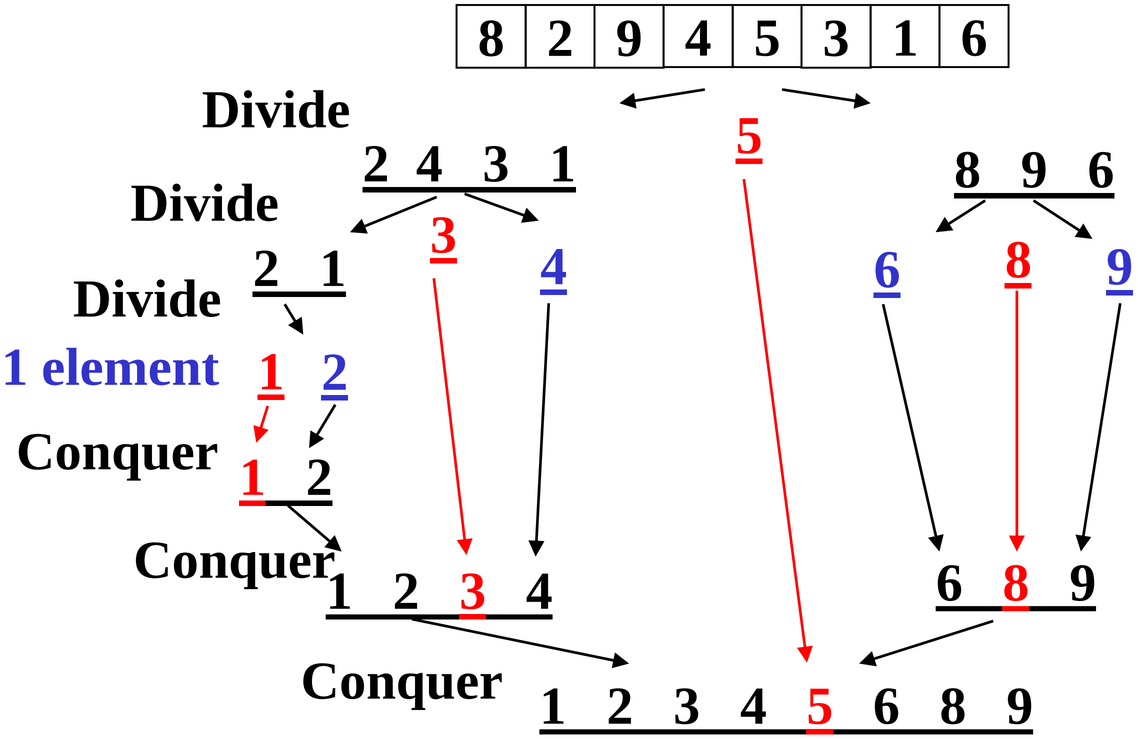
We have not yet explained:

- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about equal in size

- How to implement partitioning
  - In linear time
  - In place

Pivots

- Best pivot?
  - Median
  - Halve each time



- Worst pivot?
  - Greatest/least element
  - Reduce to problem of size 1 smaller
  - $O(n^2)$

Quicksort: Potential pivot rules

While sorting **`arr`** from **`lo`** (inclusive) to **`hi`** (exclusive)…

- Pick **`arr[lo]`** or **`arr[hi-1]`**
  - Fast, but worst-case is (mostly) sorted input

- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - (Still probably the most elegant approach)

- Median of 3, e.g., **`arr[lo], arr[hi-1], arr[(hi+lo)/2]`**
  - Common heuristic that tends to work well

Partitioning

- That is, given 8, 4, 2, 9, 3, 5, 7 and pivot 5
  – Dividing into left half & right half (based on pivot)

- Conceptually simple, but hardest part to code up correctly
  – After picking pivot, need to partition
    - Ideally in linear time
    - Ideally in place

- Ideas?

Partitioning

- One approach (there are slightly fancier ones):
  1. Swap pivot with **`arr[lo]`**; move it ‘out of the way’
  2. Use two fingers **`i`** and **`j`**, starting at **`lo+1`** and **`hi-1`** (start & end of range, apart from pivot)
  3. Move from right until we hit something less than the pivot; belongs on left side
     Move from left until we hit something greater than the pivot; belongs on right side
     Swap these two; keep moving inward

     ```
     while (i < j)
        if (arr[j] > pivot) j--
        else if (arr[i] <= pivot) i++
        else swap arr[i] with arr[j]
     ```
  4. Put pivot back in middle (Swap with **`arr[i]`**)

Quicksort Example

- Step one: pick pivot as median of 3
  - **`lo`** = 0, **`hi`** = 10

0	1	2	3	4	5	6	7	8	9
8	1	4	9	0	3	5	2	7	6

- Step two: move pivot to the **`lo`** position

0	1	2	3	4	5	6	7	8	9
6	1	4	9	0	3	5	2	7	8

Quicksort Example

Often have more than one swap during partition – this is a short example

Now partition in place

6	1	4	9	0	3	5	2	7	8

Move fingers

6	1	4	9	0	3	5	2	7	8

Swap

6	1	4	2	0	3	5	9	7	8

Move fingers

6	1	4	2	0	3	5	9	7	8

Move pivot

5	1	4	2	0	3	6	9	7	8

Quicksort Analysis

- Best-case?

- Worst-case?

- Average-case?

Quicksort Analysis

- Best-case: Pivot is always the median

  $T(0)=T(1)=1$

  $T(n)=2T(n/2) + n$ -- linear-time partition

  Same recurrence as mergesort: $O(n \log n)$

- Worst-case: Pivot is always smallest or largest element

  $T(0)=T(1)=1$

  $T(n) = 1T(n-1) + n$

  Basically same recurrence as selection sort: $O(n^2)$

- Average-case (e.g., with random pivot)
  - $O(n \log n)$, not responsible for proof (in text)

Quicksort Cutoffs

- For small *n*, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for large *n*
  - Also, recursive calls add a lot of overhead for small n
- Common engineering technique: switch to a different algorithm for subproblems below a cutoff
  - Reasonable rule of thumb: use insertion sort for $n < 10$
- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - switch to sequential algorithm
  - None of this affects asymptotic complexity

Quicksort Cutoff skeleton

```
void quicksort(int[] arr, int lo, int hi) {
  if(hi – lo < CUTOFF)
     insertionSort(arr,lo,hi);
  else
     …
}
```

Notice how this cuts out the vast majority of the recursive calls

- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree