# CSE 332: Data Structures & Parallelism

# Lecture 16: Parallel Prefix, Pack, and Sorting

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# *Outline*

Done:

- Simple ways to use parallelism for counting, summing, finding
- Analysis of running time and implications of Amdahl's Law

Now: Clever ways to parallelize more than is intuitively possible

- Parallel prefix:
  - This "key trick" typically underlies surprising parallelization
  - Enables other things like packs (aka filters)
- Parallel sorting: quicksort (not in place) and mergesort
  - Easy to get a little parallelism
  - With cleverness can get a lot

# *The prefix-sum problem*

Given **`int[] input`**, produce **`int[] output`** where:

**`output[i]`** = **`input[0]+input[1]+…+input[i]`**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **input** | 6 | 4 | 16 | 10 | 16 | 14 | 2 | 8 |
| **output** | 6 | 10 | 26 | 36 | 52 | 66 | 68 | 76 |

Sequential can be a CSE142 exam problem:

```
int[] prefix_sum(int[] input){
  int[] output = new int[input.length];
  output[0] = input[0];
  for(int i=1; i < input.length; i++)
    output[i] = output[i-1]+input[i];
  return output;
}
```

Does not seem parallelizable

- Work: $O(n)$, Span: $O(n)$
- *This algorithm* is sequential, but a *different algorithm* has Work: $O(n)$, Span: $O(\log n)$

# *Parallel prefix-sum*

- The parallel-prefix algorithm does two passes
  - Each pass has $O(n)$ work and $O(\mathtt{log}\ n)$ span
  - So in total there is $O(n)$ work and $O(\mathtt{log}\ n)$ span
  - So like with array summing, parallelism is $n/\mathtt{log}\ n$
    - An exponential speedup
- First pass builds a tree bottom-up: the “up” pass
- Second pass traverses the tree top-down: the “down” pass

# *Local bragging*

Historical note:

- Original algorithm due to R. Ladner and M. Fischer at UW in 1977
- Richard Ladner joined the UW faculty in 1971 and hasn’t left

1968? 1973?

recent

# *Parallel Prefix: The Up Pass*

We build want to build a binary tree where

- Root has sum of the range [x,y)
- If a node has sum of [lo,hi) and hi>lo,
  - Left child has sum of [lo,middle)
  - Right child has sum of [middle,hi)
  - A leaf has sum of [i,i+1), which is simply input[i]

It is critical that we actually create the tree as we will need it for the down pass

- We do not need an actual linked structure
- We could use an array as we did with heaps

Analysis of first step: Work = Span =

# *The algorithm, part 1*

Specifically…..

1. Propagate ‘sum’ up: Build a binary tree where
   - Root has sum of **`input[0]..input[n-1]`**
   - Each node has sum of **`input[lo]..input[hi-1]`**
     - Build up from leaves; parent.sum=left.sum+right.sum
   - A leaf’s sum is just it’s value; **`input[i]`**

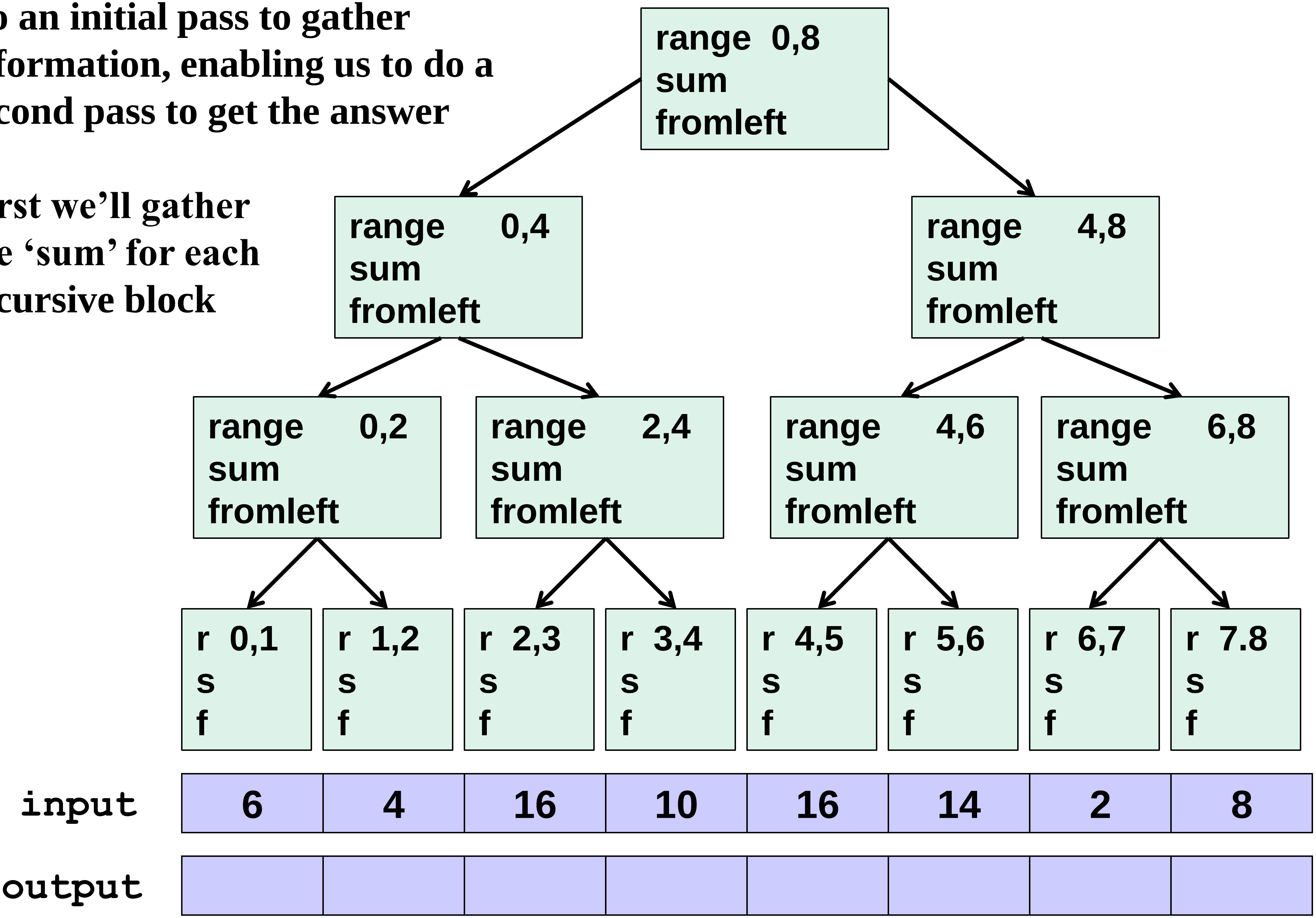
This is an easy fork-join computation: combine results by actually building a binary tree with all the sums of ranges

- Tree built bottom-up in parallel
- Could be more clever; ex. Use an array as tree representation like we did for heaps

Analysis of first step: $O(n)$ **work**, $O(\log n)$ **span**

**The (completely non-obvious) idea:**
**Do an initial pass to gather information, enabling us to do a second pass to get the answer**

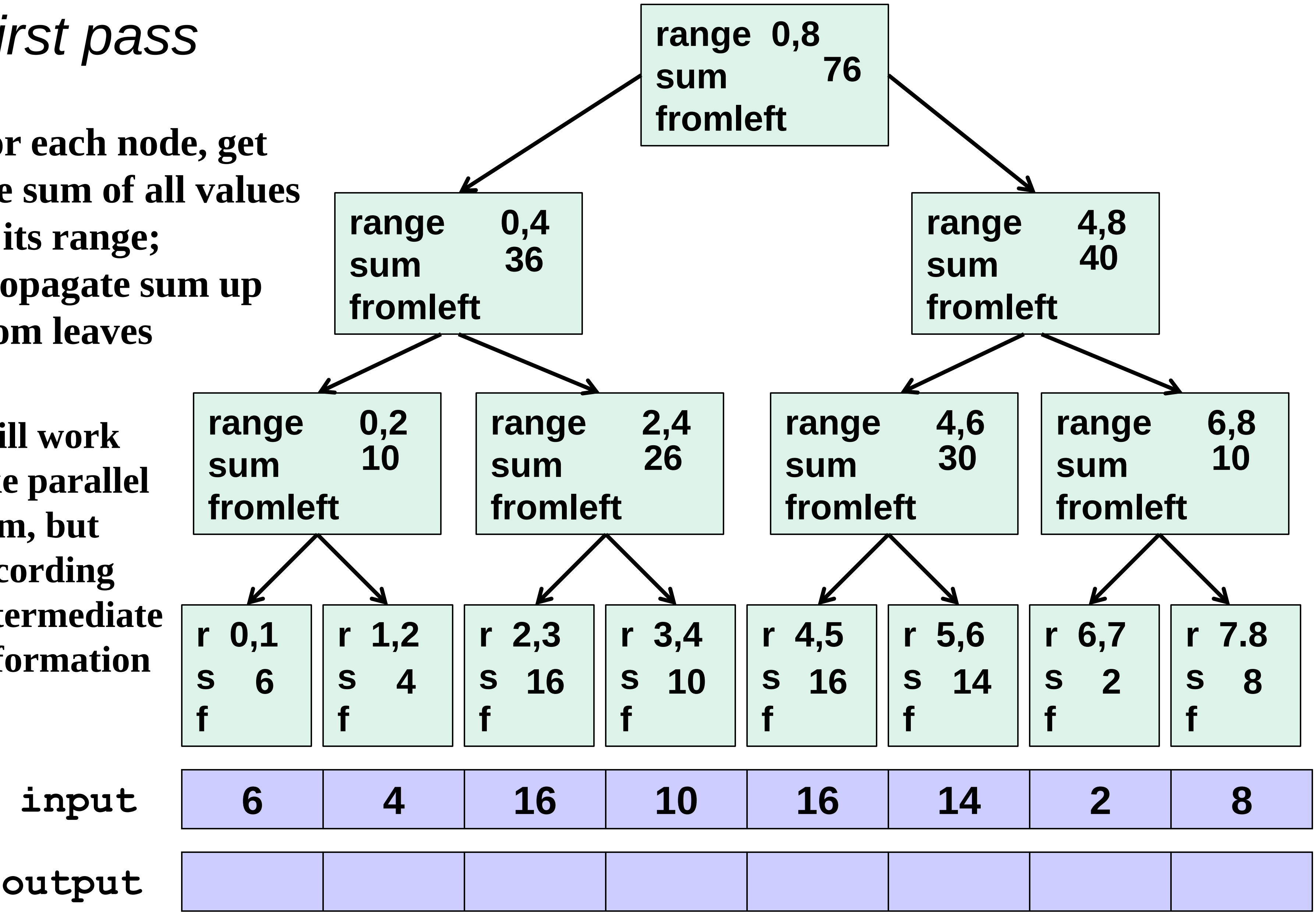
**First we'll gather the 'sum' for each recursive block**



# *First pass*

**For each node, get the sum of all values in its range; propagate sum up from leaves**

**Will work like parallel sum, but recording intermediate information**



# *The algorithm, part 2*

2. Propagate ‘fromleft’ down:
   - Root given a **`fromLeft`** of **`0`**
   - Node takes its **`fromLeft`** value and
     - Passes its left child the same **`fromLeft`**
     - Passes its right child its **`fromLeft`** plus its left child’s **`sum`** (as stored in part 1)
   - At the leaf for array position **`i`**, **`output[i]=fromLeft+input[i]`**

This is an easy fork-join computation: traverse the tree built in step 1 and produce no result (the leaves assign to **`output`**)

- Invariant: **`fromLeft`** is sum of elements left of the node’s range

Analysis of first step: $O(n)$ **work**, $O($**`log`** $n)$ **span**

Analysis of second step:

**Total for algorithm:**

# *The algorithm, part 2*

2. Propagate ‘fromleft’ down:
   - Root given a **`fromLeft`** of **`0`**
   - Node takes its **`fromLeft`** value and
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This is an easy fork-join computation: traverse the tree built in step 1 and produce no result (the leaves assign to **`output`**)

- Invariant: **`fromLeft`** is sum of elements left of the node’s range

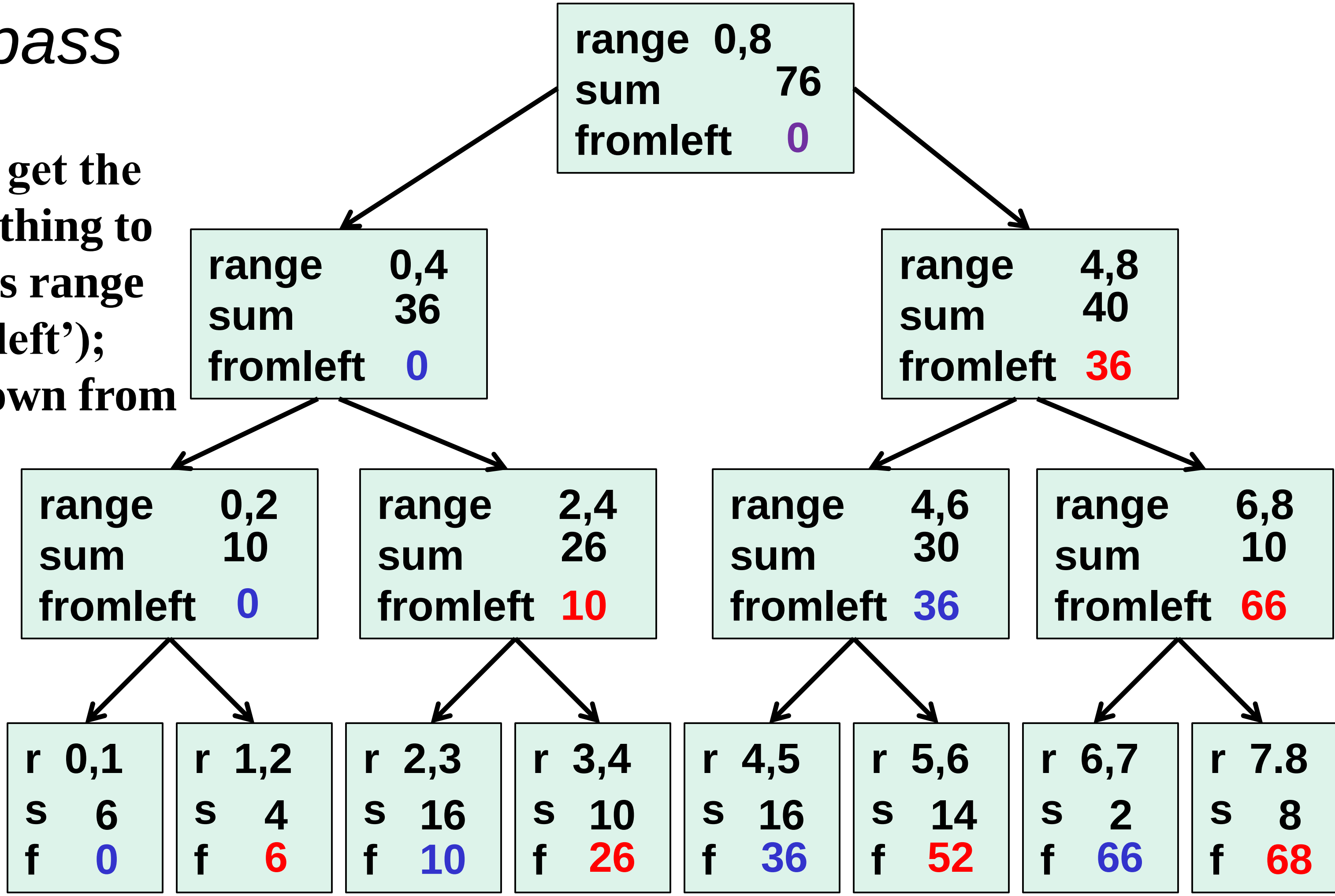
Analysis of first step: $O(n)$ **work**, $O(\texttt{log}\ n)$ **span**

Analysis of second step: $O(n)$ **work**, $O(\texttt{log}\ n)$ **span**

**Total for algorithm:** $O(n)$ **work**, $O(\texttt{log}\ n)$ **span**

# *Second pass*

**Using 'sum', get the sum of everything to the left of this range (call it 'fromleft'); propagate down from root**




| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| `input` | 6 | 4 | 16 | 10 | 16 | 14 | 2 | 8 |
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# *Sequential cut-off*

Adding a sequential cut-off isn't too bad:

- **Step One**: Propagating Up the **`sum`**s:
  - Have a leaf node just hold the sum of a range of values instead of just one array value (Sequentially compute sum for that range)
  - The tree itself will be shallower
- **Step Two**: Propagating Down the **`fromLeft`**s:
  - Have leaf compute prefix sum sequentially over its [lo,hi):

```
output[lo] = fromLeft + input[lo];
for(i=lo+1; i < hi; i++)
  output[i] = output[i-1] + input[i]
```

# *Parallel prefix, generalized*

Just as sum-array was the simplest example of a common pattern, prefix-sum illustrates a pattern that arises in many, many problems

- Minimum, maximum of all elements to the left of `i`
- Is there an element to the left of `i` satisfying some property?
- Count of elements to the left of `i` satisfying some property
  - This last one is perfect for an efficient parallel pack…
  - Perfect for building on top of the "parallel prefix trick"

# *Pack (think "Filter")*

[Non-standard terminology]

Given an array **`input`**, produce an array **`output`** containing <u>only</u> elements such that **`f(element)`** is **`true`**

Example: **`input [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]`**

**`f: "is element > 10"`**

**`output [17, 11, 13, 19, 24]`**

Parallelizable?

- Determining *<u>whether</u>* an element belongs in the output is easy
- But determining *<u>where</u>* an element belongs in the output is hard; seems to depend on previous results….

In this example, Filter = element > 10

# *Parallel Pack = (Soln)*
# *parallel map + parallel prefix + parallel map*

1. **Parallel map** to compute a bit-vector for true elements:

```
input  [17, 4, 6, 8, 11, 5, 13, 19, 0, 24]
bits   [1,  0, 0, 0,  1, 0,  1,  1, 0,  1]
```

2. **Parallel-prefix** sum ***on the bit-vector***:

```
bitsum [1,  1, 1, 1,  2, 2,  3,  4, 4,  5]
```

3. **Parallel map** to produce the output:

```
output [17, 11, 13, 19, 24]
```

```
output = new array of size bitsum[n-1]
FORALL(i=0; i < input.length; i++){


}
```

# *Pack comments*

- First two steps can be combined into one pass
  - Just using a different base case for the prefix sum
  - No effect on asymptotic complexity

- Can also combine third step into the down pass of the prefix sum
  - Again no effect on asymptotic complexity

- Analysis: $O(n)$ work, $O(\texttt{log}\ n)$ span
  - 2 or 3 passes, but 3 is a constant ☺

- Parallelized packs will help us parallelize quicksort…

# *Sequential Quicksort review*

Recall quicksort was sequential, in-place, expected time $O(n \; \mathtt{log} \; n)$

| | **Best / expected case *work*** |
|---|---|
| 1. **Pick a pivot element** | **O(1)** |
| 2. **Partition all the data into:** | **O(n)** |
| A. **The elements less than the pivot** | |
| B. **The pivot** | |
| C. **The elements greater than the pivot** | |
| 3. **Recursively sort A and C** | **2T(n/2)** |

Recurrence (assuming a good pivot):

T(0)=T(1)=1

T(n)=__________________

Run-time: O(nlogn)

How should we parallelize this?

# *Review: Really common recurrences*

Should know how to solve recurrences but also recognize some really common ones:

| | |
|---|---|
| $T(n) = O(1) + T(n\text{-}1)$ | linear |
| $T(n) = O(1) + 2T(n/2)$ | linear |
| $T(n) = O(1) + T(n/2)$ | logarithmic |
| $T(n) = O(1) + 2T(n\text{-}1)$ | exponential |
| $T(n) = O(n) + T(n\text{-}1)$ | quadratic |
| $T(n) = O(n) + T(n/2)$ | linear |
| $T(n) = O(n) + 2T(n/2)$ | $O$(n `log` n) |

Note big-Oh can also use more than one variable

- Example: can sum all elements of an *n*-by-*m* matrix in $O(nm)$

# *Parallel Quicksort (version 1)*

| | **Best / expected case *work*** |
|---|---|
| **1. Pick a pivot element** | **O(1)** |
| **2. Partition all the data into:** | **O(n)** |
| **A. The elements less than the pivot** | |
| **B. The pivot** | |
| **C. The elements greater than the pivot** | |
| **3. Recursively sort A and C** | **2T(n/2)** |

First: Do the two recursive calls in parallel

- **Work**:
- **Span**: now recurrence takes the form:

**Span**:

# *Doing better*

- $O($`log` $n)$ speed-up with an infinite number of processors is okay, but a bit underwhelming
  - Sort $10^9$ elements 30 times faster

- Google searches strongly suggest quicksort cannot do better because the partition cannot be parallelized
  - The Internet has been known to be wrong ☺
  - But we need auxiliary storage (no longer in place)
  - In practice, constant factors may make it not worth it, but remember Amdahl's Law…(exposing parallelism is important!)

- Already have everything we need to parallelize the partition…

# *Parallel partition (not in place)*

**Partition all the data into:**

A. **The elements less than the pivot**
B. **The pivot**
C. **The elements greater than the pivot**

- This is just two packs!
  - We know a pack is $O(n)$ work, $O(\texttt{log}\ n)$ span
  - Pack elements less than pivot into left side of **`aux`** array
  - Pack elements greater than pivot into right size of **`aux`** array
  - Put pivot between them and recursively sort
  - With a little more cleverness, can do both packs at once but no effect on asymptotic complexity
- With _________ span for partition, the total span for quicksort is
  $T(n) =$

# *Parallel Quicksort Example (version 2)*

- Step 1: pick pivot as median of three

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 8 | 1 | 4 | 9 | 0 | 3 | 5 | 2 | 7 | 6 |

- Steps 2a and 2c (combinable): pack less than, then pack greater than into a second array
  - Fancy parallel prefix to pull this off (not shown)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 3 | 5 | 2 | | | | |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 4 | 0 | 3 | 5 | 2 | 6 | 8 | 9 | 7 |

- Step 3: Two recursive sorts in parallel
  - Can sort back into original array (like in mergesort)

# *Parallelize Mergesort?*

Recall mergesort: sequential, **not**-in-place, worst-case $O(n \log n)$

1. **Sort left half and right half** **2T(n/2)**
2. **Merge results** **O(n)**

Just like quicksort, doing the two recursive sorts in parallel changes the recurrence for the **Span** to $T(n) = O(n) + 1T(n/2) = O(n)$

- Again, **Work** is O(nlogn), and
- parallelism is work/span = $O(\mathbf{log}\ n)$
- To do better, *need to parallelize the merge*
  - The trick won't use parallel prefix this time…

# *Parallelizing the merge*

Need to merge two *sorted* subarrays (may not have the same size)

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 4 | 8 | 9 | | 2 | 3 | 5 | 6 | 7 |

**Idea**: Suppose the larger subarray has $m$ elements. In parallel:

- Merge the first $m/2$ elements of the larger half with the "appropriate" elements of the smaller half
- Merge the second $m/2$ elements of the larger half with the rest of the smaller half

# *Parallelizing the merge (in more detail)*

Need to merge two ***sorted*** subarrays (may not have the same size)

**Idea**: Recursively divide subarrays in half, merge halves in parallel

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 4 | 6 | 8 | 9 | | 1 | 2 | 3 | 5 | 7 |

Suppose the larger subarray has *m* elements. In parallel:

- Pick the **median** element of the larger array (here 6) in constant time
- In the other array, use binary search to find the first element greater than or equal to that median (here 7)

Then, in parallel:

- Merge half the larger array (from the median onward) with the upper part of the shorter array
- Merge the lower part of the larger array with the lower part of the shorter array

# *Example: Parallelizing the Merge*

| | | | | |
|---|---|---|---|---|
| **0** | **4** | **6** | **8** | **9** |

| | | | | |
|---|---|---|---|---|
| **1** | **2** | **3** | **5** | **7** |

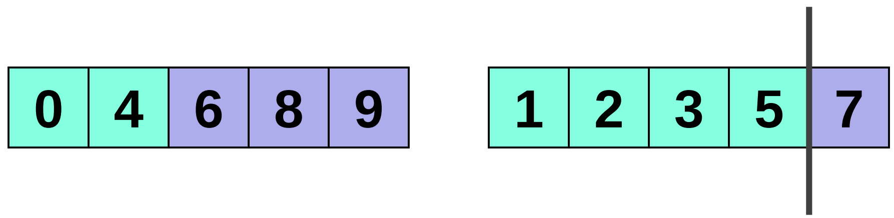
# *Example: Parallelizing the Merge*

| | | | | |
|---|---|---|---|---|
| **0** | **4** | **6** | **8** | **9** |

| | | | | |
|---|---|---|---|---|
| **1** | **2** | **3** | **5** | **7** |

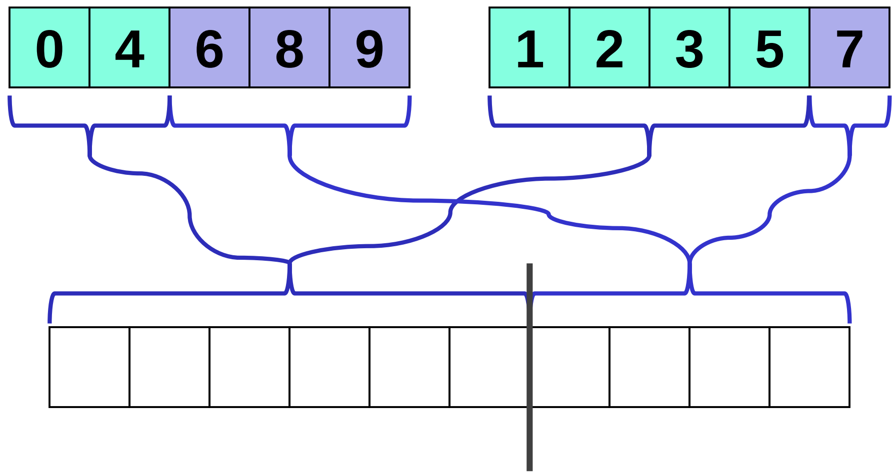
1. Get median of bigger half: ***O*(1) to compute middle index**

# *Example: Parallelizing the Merge*



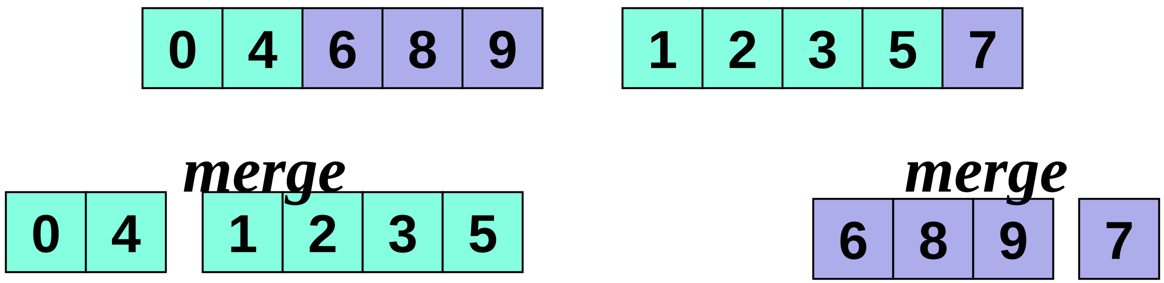

1. Get median of bigger half: ***O*(1) to compute middle index**
2. **Find how to split the smaller half at the same value: O(`log` n) to do binary search on the sorted small half**

# *Example: Parallelizing the Merge*



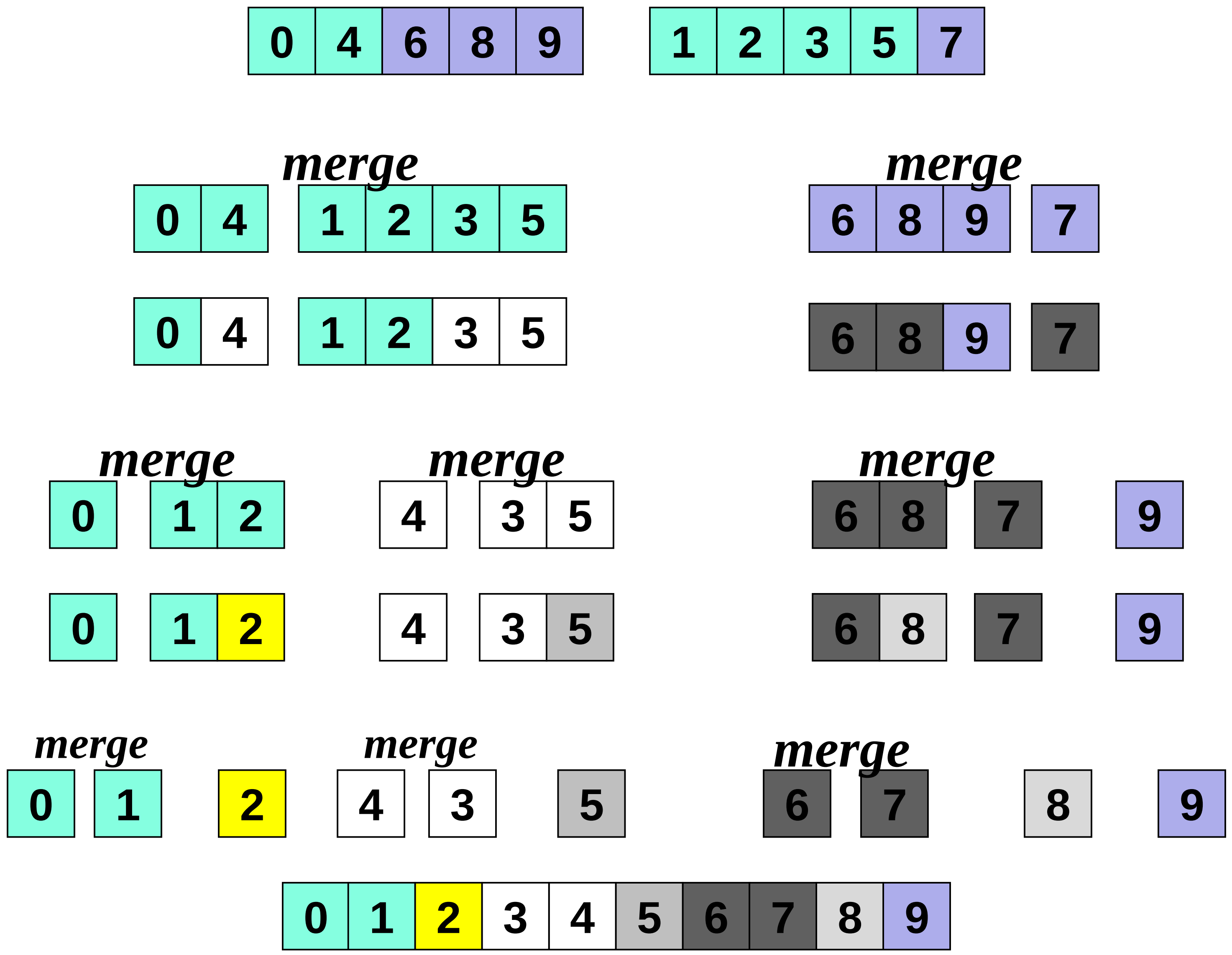

1. Get median of bigger half: ***O*(1) to compute middle index**
2. **Find how to split the smaller half at the same value: O(`log` n) to do binary search on the sorted small half**
3. **Size of two sub-merges conceptually splits output array: *O*(1)**

# *Example: Parallelizing the Merge*

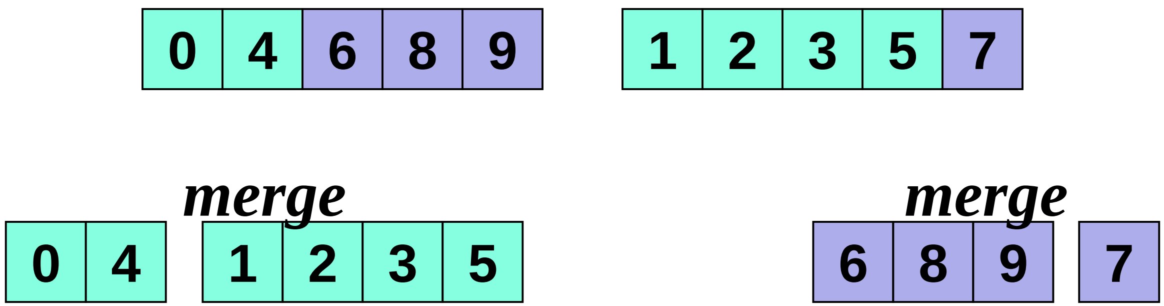



1. Get median of bigger half: ***O*(1) to compute middle index**
2. **Find how to split the smaller half at the same value:**
   **O(`log` n) to do binary search on the sorted small half**
3. **Two sub-merges conceptually splits output array: *O*(1)**
4. **Do two submerges in parallel**

# Example: Parallelizing the Merge

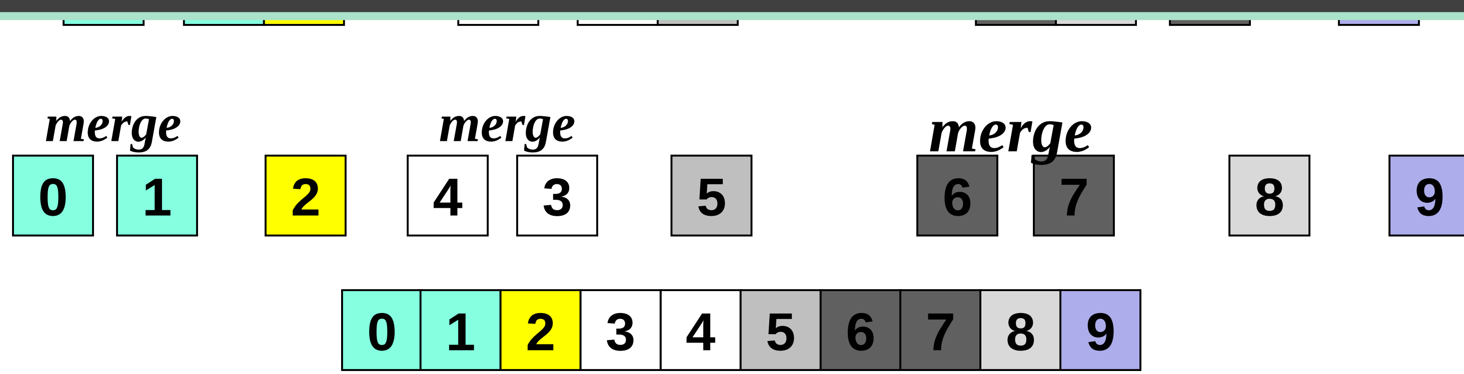


# Example: Parallelizing the Merge




**When we do each merge in parallel:**

- **we split the bigger array in half**
- **use binary search to split the smaller array**
- **And in base case we do the copy**



# *Parallel Merge Pseudocode*

Merge(arr[], $left_1$, $left_2$, $right_1$, $right_2$, out[], $out_1$, $out_2$ )

```
int leftSize = left₂ – left₁
int rightSize = right₂ – right₁
// Assert: out₂ – out₁ = leftSize + rightSize
// We will assume leftSize > rightSize without loss of generality

if (leftSize + rightSize < CUTOFF)
    sequential merge and copy into out[out1..out2]

int mid = (left₂ – left₁)/2
binarySearch arr[right1..right2] to find j such that
    arr[j] ≤ arr[mid] ≤ arr[j+1]

Merge(arr[], left₁, mid, right₁, j, out[], out₁, out₁+mid+j)
Merge(arr[], mid+1, left₂, j+1, right₂, out[], out₁+mid+j+1, out₂)
```

# *Analysis*

- *Sequential* mergesort:

  T($n$) = 2T($n$/2) + $O$($n$)     which is $O$($n$ **`log`** $n$)

- Doing the *two recursive calls in parallel* but a sequential merge:

  **Work**: same as sequential

  **Span**: T($n$)=**1**T($n$/2)+$O$($n$)     which is $O$($n$)

- *Parallel merge* makes **work** and **span** harder to compute…
  – Each merge step does an extra $O$(**`log`** $n$) binary search to find how to split the smaller subarray
  – To merge $n$ elements total, do two smaller merges of possibly different sizes
  – But worst-case split is (3/4)$n$ and (1/4)$n$
    - Happens when the two subarrays are of the same size (n/2) and the "smaller" subarray splits into two pieces of the most uneven sizes possible: one of size n/2, one of size 0

"larger"

| | | | |
|---|---|---|---|
| **0** | **4** | **6** | **8** |

"smaller"

| | | | |
|---|---|---|---|
| **1** | **2** | **3** | **5** |

# *Analysis continued*

For **<u>just</u>** a parallel merge of $n$ elements:

- **Work** is $T(n) = T(3n/4) + T(n/4) + O(\mathtt{log}\ n)$ which is $O(n)$
- **Span** is $T(n) = T(3n/4) + O(\mathtt{log}\ n)$, which is $O(\mathtt{log}^2\ n)$
- (neither bound is immediately obvious, but “trust me”)

So for **<u>mergesort</u>** with <u>*parallel merge*</u> overall:

- **Work** is $T(n) = 2T(n/2) + O(n)$, which is $O(n\ \mathtt{log}\ n)$
- **Span** is $T(n) = 1T(n/2) + O(\mathtt{log}^2\ n)$, which is $O(\mathtt{log}^3\ n)$

So parallelism (work / span) is $O(n / \mathtt{log}^2\ n)$

- Not quite as good as quicksort’s $O(n / \mathtt{log}\ n)$
  - But (unlike Quicksort) this is a worst-case guarantee
- And as always this is just the asymptotic result