CSE 332: Data Structures & Parallelism

Lecture 11:More Hashing

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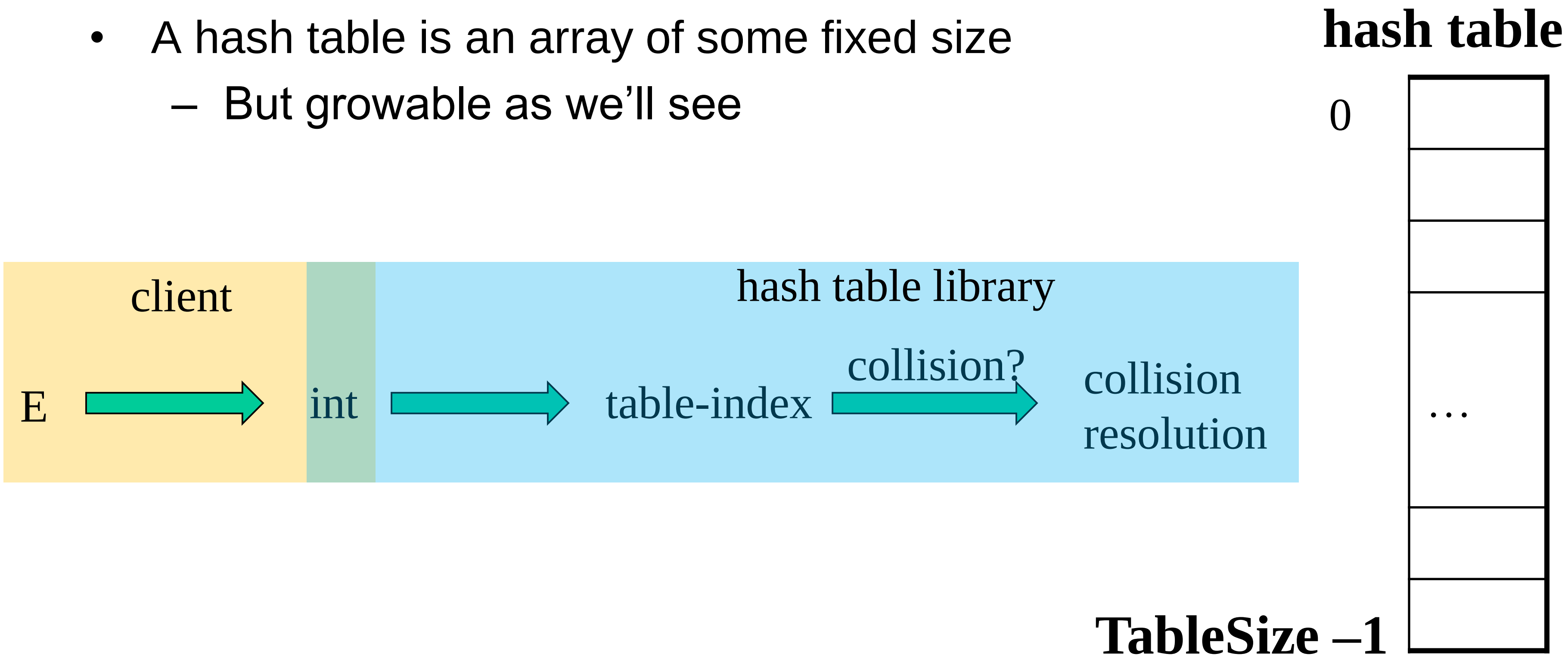
Today

- Dictionaries
  - Hashing

Hash Tables: Review

- Aim for constant-time (i.e., $O(1)$) **`find`**, **`insert`**, and **`delete`**
  - "On average" under some reasonable assumptions
- A hash table is an array of some fixed size
  - But growable as we'll see



Hashing Choices

1. Choose a Hash function
2. Choose TableSize
3. Choose a Collision Resolution Strategy from these:
   - Separate Chaining
   - Open Addressing
     - Linear Probing
     - Quadratic Probing
     - Double Hashing

- Other issues to consider:
  - Deletion?
  - What to do when the hash table gets “too full”?

Open Addressing: Linear Probing

- Why not use up the empty space in the table?
- Store directly in the array cell (no linked list)
- How to deal with collisions?
- If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full…

- Example: insert 38, 19, 8, 109, 10

0	
1	
2	
3	
4	
5	
6	
7	
8	38
9	

Open Addressing: Linear Probing

- Another simple idea: If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full…

- Example: insert 38, 19, 8, 109, 10

0	/
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open Addressing: Linear Probing

- Another simple idea: If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full…

- Example: insert 38, 19, 8, 109, 10

0	8
1	/
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open Addressing: Linear Probing

- Another simple idea: If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full…

- Example: insert 38, 19, 8, 109, 10

0	8
1	109
2	/
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open Addressing: Linear Probing

- Another simple idea: If **`h(key)`** is already full,
  - try **`(h(key) + 1) % TableSize`**. If full,
  - try **`(h(key) + 2) % TableSize`**. If full,
  - try **`(h(key) + 3) % TableSize`**. If full…

- Example: insert 38, 19, 8, 109, 10

0	8
1	109
2	10
3	/
4	/
5	/
6	/
7	/
8	38
9	19

Open addressing

Linear probing is *one example* of open addressing

In general, open addressing means resolving collisions by trying a sequence of other positions in the table.

Trying the *next* spot is called probing

- We just did linear probing:
  - **`i`**th probe: **`(h(key) + i) % TableSize`**
- In general have some probe function **`f`** and :
  - **`i`**th probe: **`(h(key) + f(i)) % TableSize`**

Open addressing does poorly with high load factor λ

- So want larger tables
- Too many probes means no more *O*(1)

Terminology

We and the book use the terms

- “chaining” or “separate chaining”
- “open addressing”

Very confusingly,

- “open hashing” is a synonym for “chaining”
- “closed hashing” is a synonym for “open addressing”

Open Addressing: Linear Probing

What about **`find`**? If value is in table? If not there? Worst case?

What about **`delete`**?

How does open addressing with linear probing compare to separate chaining?

Open Addressing: Other Operations

`insert` finds an open table position using a probe function

What about **`find`**?

- Must use same probe function to "retrace the trail" for the data
- Unsuccessful search when reach empty position

What about **`delete`**?

- ***Must*** use "lazy" deletion. Why?
  - Marker indicates "no data here, but don't stop probing"

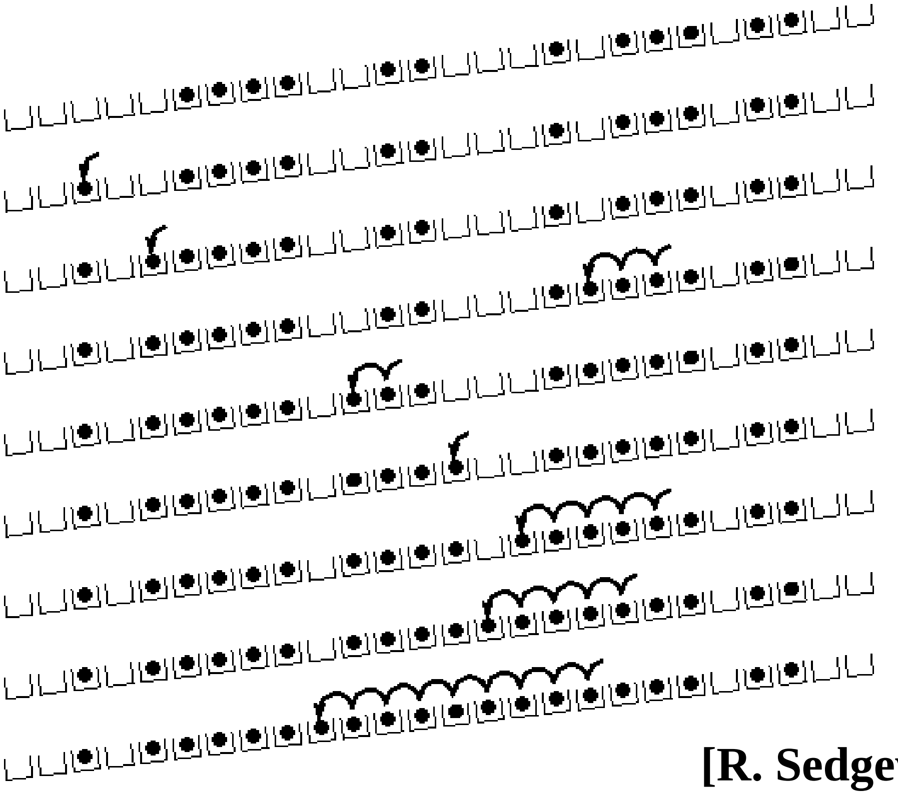
10	✘	/	23	/	/	16	✘	26

  - As with lazy deletion on other data structures, on insert, spots marked "deleted" can be filled in.
- Note: **`delete`** with chaining is plain-old list-remove

Primary Clustering

It turns out linear probing is a *bad idea*, even though the probe function is quick to compute (a good thing)

- Tends to produce *clusters*, which lead to long probe sequences
- Called primary clustering
- Saw the start of a cluster in our linear probing example



[R. Sedgewick]

Analysis of Linear Probing

- **Trivial fact:** For any $\lambda < 1$, linear probing will find an empty slot
  - It is "safe" in this sense: no infinite loop unless table is full

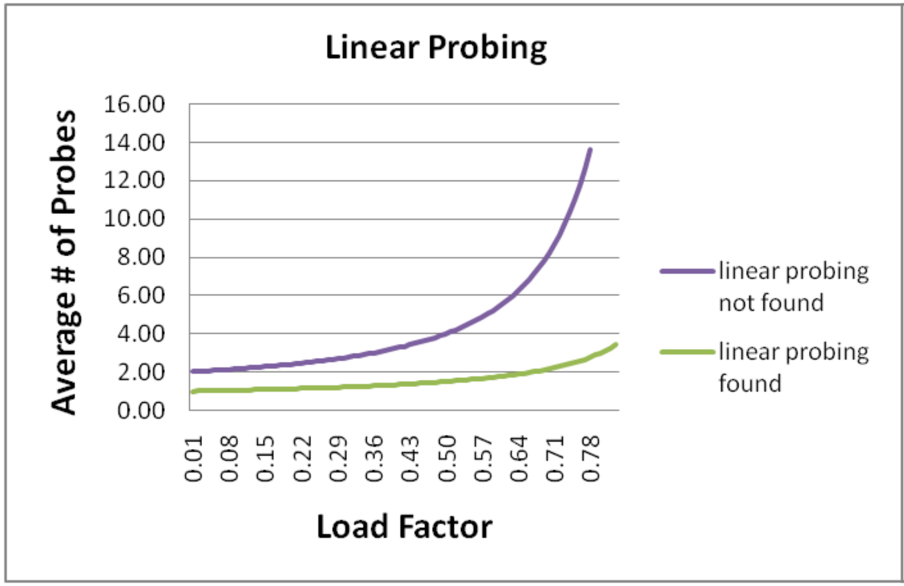
- **Non-trivial facts** we won't prove:

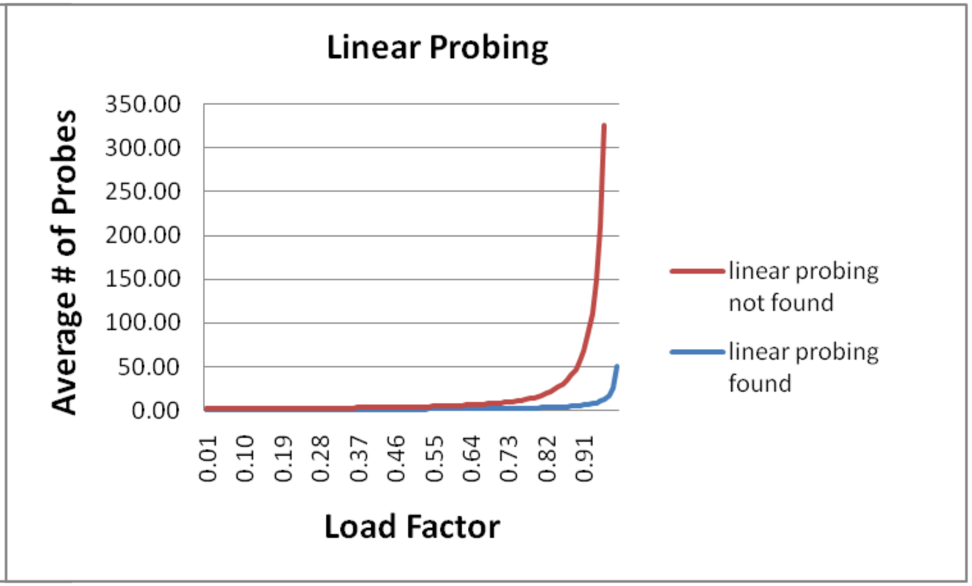
  Average # of probes given λ (in the limit as `TableSize` $\rightarrow \infty$)
  - Unsuccessful search: $$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$$
  - Successful search: $$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$$

- This is pretty bad: need to leave sufficient empty space in the table to get decent performance (see chart)

Analysis in chart form

- Linear-probing performance degrades rapidly as table gets full
  - (Formula assumes “large table” but point remains)





- By comparison, separate chaining performance is linear in λ and has no trouble with $\boldsymbol{\lambda>1}$

Open Addressing: Linear probing

`(h(key) + f(i)) % TableSize`

- For linear probing:

  `f(i) = i`

- So probe sequence is:
  - 0^{th} probe: **`h(key) % TableSize`**
  - 1^{st} probe: **`(h(key) + 1) % TableSize`**
  - 2^{nd} probe: **`(h(key) + 2) % TableSize`**
  - 3^{rd} probe: **`(h(key) + 3) % TableSize`**
  - …
  - i^{th} probe: **`(h(key) + i) % TableSize`**

Open Addressing: Quadratic probing

- We can avoid primary clustering by changing the probe function…

  `(h(key) + f(i)) % TableSize`

  – For quadratic probing:

    `f(i) =` i^2

  – So probe sequence is:
    - 0^{th} probe: **`h(key) % TableSize`**
    - 1^{st} probe: **`(h(key) + 1) % TableSize`**
    - 2^{nd} probe: **`(h(key) + 4) % TableSize`**
    - 3^{rd} probe: **`(h(key) + 9) % TableSize`**
    - …
    - i^{th} probe: **`(h(key) +`** i^2**`) % TableSize`**

- Intuition: Probes quickly “leave the neighborhood”

ith probe: (`h(key) + i`^{2}`) % TableSize`

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

TableSize=10

Insert:

89

18

49

58

79

Quadratic Probing Example

TableSize = 10

insert(89)

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	89

TableSize = 10

insert(89)

insert(18)

Quadratic Probing Example

TableSize = 10

insert(89)

insert(18)

insert(49)

0	
1	
2	
3	
4	
5	
6	
7	
8	**18**
9	89

Quadratic Probing Example

0	49
1	
2	
3	
4	
5	
6	
7	
8	18
9	89

TableSize = 10

insert(89)

insert(18)

insert(49)

> **49 % 10 = 9 collision!**
>
> **(49 + 1) % 10 = 0**

insert(58)

Quadratic Probing Example

0	49
1	
2	58
3	
4	
5	
6	
7	
8	18
9	89

TableSize = 10

insert(89)

insert(18)

insert(49)

insert(58)

> **58 % 10 = 8 collision!**
>
> **(58 + 1) % 10 = 9 collision!**
>
> **(58 + 4) % 10 = 2**

insert(79)

Quadratic Probing Example

0	49
1	
2	58
3	79
4	
5	
6	
7	
8	18
9	89

TableSize = 10

insert(89)

insert(18)

insert(49)

insert(58)

insert(79)

79 % 10 = 9 collision!

(79 + 1) % 10 = 0 collision!

(79 + 4) % 10 = 3

ith probe: (`h(key)` + i^2) % `TableSize`

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: (`h(key)` + i^2) % `TableSize`

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: (`h(key) + i`^{2}`) % TableSize`

Another Quadratic Probing Example

0	
1	
2	
3	
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: (`h(key) + i`^{2}`) % TableSize`

Another Quadratic Probing Example

0	48
1	
2	
3	
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

Another Quadratic Probing Example

0	48
1	
2	5
3	
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: (`h(key) + i`^{2}`) % TableSize`

Another Quadratic Probing Example

0	48
1	
2	5
3	55
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

ith probe: (`h(key) + i`^{2}`) % TableSize`

Another Quadratic Probing Example

0	48
1	
2	5
3	55
4	
5	40
6	76

TableSize = 7

Insert:

76	**(76 % 7 = 6)**
40	**(40 % 7 = 5)**
48	**(48 % 7 = 6)**
5	**(5 % 7 = 5)**
55	**(55 % 7 = 6)**
47	**(47 % 7 = 5)**

(47 + 1) % 7 = 6 collision!

(47 + 4) % 7 = 2 collision!

(47 + 9) % 7 = 0 collision!

(47 + 16) % 7 = 0 collision!

(47 + 25) % 7 = 2 collision!

Will we ever get a 1 or 4?!?

Another Quadratic Probing Example

0	48
1	
2	5
3	55
4	
5	40
6	76

insert(47) will always fail here. Why?

For all *i*, $(5 + i^2) \% 7$ is 0, 2, 5, or 6

Proof uses induction and

$$(5 + i^2) \% 7 = (5 + (i - 7)^2) \% 7$$

In fact, for all *c* and *k*,

$$(c + i^2) \% k = (c + (i - k)^2) \% k$$

From bad news to good news

Bad News:

- After **`TableSize`** quadratic probes, we cycle through the same indices

Good News:

- If **`TableSize`** is *prime* and λ < ½, then quadratic probing will find an empty slot in at most **`TableSize/2`** probes
- So: If you keep λ < ½ and **`TableSize`** is *prime*, no need to detect cycles
- Proof posted in **`lecture11.txt`** (slightly less detailed proof in textbook)

For prime **`TableSize`** and $\mathbf{0 \le i,j \le TableSize/2}$ where $\mathbf{i \ne j}$,

$$\mathbf{(h(key) + i^2)\ \%\ TableSize \ne (h(key) + j^2)\ \%\ TableSize}$$

That is, if **`TableSize`** is prime, the first **`TableSize`**/2 quadratic probes map to different locations (and one of those will be empty if the table is < half full).

Quadratic Probing: Success guarantee for $\lambda < ½$

First size/2 probes distinct. If < half full, one is empty.

- If size is prime and $\lambda < ½$, then quadratic probing will find an empty slot in size/2 probes or fewer.
  - show for all `0 ≤ i,j ≤ size/2` and `i ≠ j`

    `(h(x) + i²) mod size ≠ (h(x) + j²) mod size`
  - by contradiction: suppose that for some i ≠ j:

    `(h(x) + i²) mod size = (h(x) + j²) mod size`

    `⇒ i² mod size = j² mod size`

    `⇒ (i² - j²) mod size = 0`

    `⇒ [(i + j)(i - j)] mod size = 0`

  BUT size does not divide **`(i-j)`** or **`(i+j)`**

ith probe and jth probe

First size/2 probes will be distinct, and if less than half of table is full then after size/2 probes you will find one of those empty spots

How can **`i+j = 0`** or **`i+j = size`** when:

`i ≠ j` and **`0 ≤ i,j ≤ size/2`**?

Similarly how can **`i-j = 0`** or **`i-j = size ?`**

One of these must be = 0 when mod size

Size would need to divide one of these

Clustering reconsidered

- Quadratic probing does not suffer from primary clustering:
As we resolve collisions we are not merely growing “big blobs” by adding one more item to the end of a cluster, we are looking i^2 locations away, for the next possible spot.

- But quadratic probing does not help resolve collisions between keys that initially hash *to the same* ***index***
  - Any 2 keys that initially hash to the same index **will have the same series of moves after that** looking for any empty spot
  - Called secondary clustering

- Can avoid secondary clustering with *a probe function that depends on the key*: double hashing…

Open Addressing: Double hashing

Idea: Given two good hash functions *h* and *g*, it is very unlikely that for some *key*, **`h(key) == g(key)`**

`(h(key) + f(i)) % TableSize`

- For double hashing:

  `f(i) = i*g(key)`

- So probe sequence is:
  - 0^{th} probe: **`h(key) % TableSize`**
  - 1^{st} probe: **`(h(key) + g(key)) % TableSize`**
  - 2^{nd} probe: **`(h(key) + 2*g(key)) % TableSize`**
  - 3^{rd} probe: **`(h(key) + 3*g(key)) % TableSize`**
  - …
  - i^{th} probe: **`(h(key) + i*g(key)) % TableSize`**

- Detail: Make sure **`g(key)`** can't be **`0`**

ith probe: `(h(key) + i*g(key)) % TableSize`

Open Addressing: Double Hashing

0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33

147

43

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	
8	
9	

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33

147

43

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	
8	**28**
9	

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33

147

43

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	**33**
8	28
9	

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33 → g(33) = 1 + 3 mod 9 = 4

147

43

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	33
8	28
9	**147**

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33

147 → g(147) = 1 + 14 mod 9 = 6

43

ith probe: `(h(key) + i*g(key)) % TableSize`

Double Hashing

0	
1	
2	
3	13
4	
5	
6	
7	33
8	28
9	147

T = 10 (TableSize)

Hash Functions:

h(key) = key mod T

g(key) = 1 + ((key/T) mod (T-1))

Insert these values into the hash table in this order. Resolve any collisions with double hashing:

13

28

33

147 → g(147) = 1 + 14 mod 9 = 6

43 → g(43) = 1 + 4 mod 9 = 5

We have a problem:

3 + 0 = 3 3 + **5** = **8** **3** + 10 = **13**

3 + **15** = **18** 3 + 20 = 23

Double-hashing analysis

- **Intuition**: Since each probe is “jumping” by **`g(key)`** each time, we “leave the neighborhood” *and* “go different places from other initial collisions”

But, as in quadratic probing, we could still have a problem where we are not "safe" due to an infinite loop despite room in table

- It is known that this cannot happen in at least one case:

For primes p and q such that $2 < q < p$

h(key) = key % p

g(key) = q – (key % q)

Yet another reason to use a prime TableSize

- So, for double hashing

  i^{th} probe: (`h(key) + i*g(key))% TableSize`
- Say g(key) divides Tablesize
  - That is, there is some integer x such that x*g(key)=Tablesize
  - After x probes, we'll be back to trying the same indices as before
- Ex:
  - Tablesize=50
  - g(key)=25
  - Probing sequence:
    - h(key)
    - h(key)+25
    - h(key)+50=h(key)
    - h(key)+75=h(key)+25
- Only 1 & itself divide a prime

More double-hashing facts

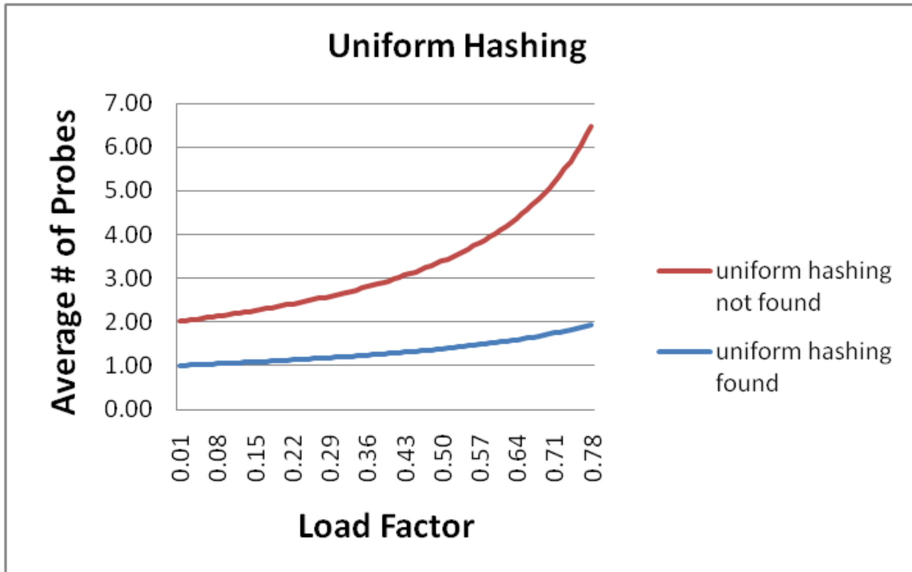
- Assume "uniform hashing"
  - Means probability of **`g(key1) % p == g(key2) % p`** is **`1/p`**

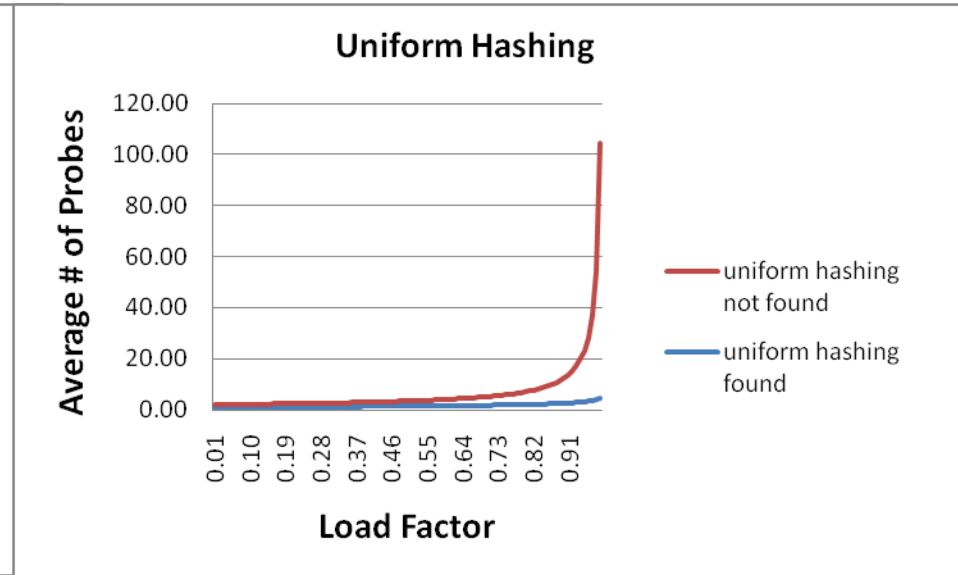
- Non-trivial facts we won't prove:

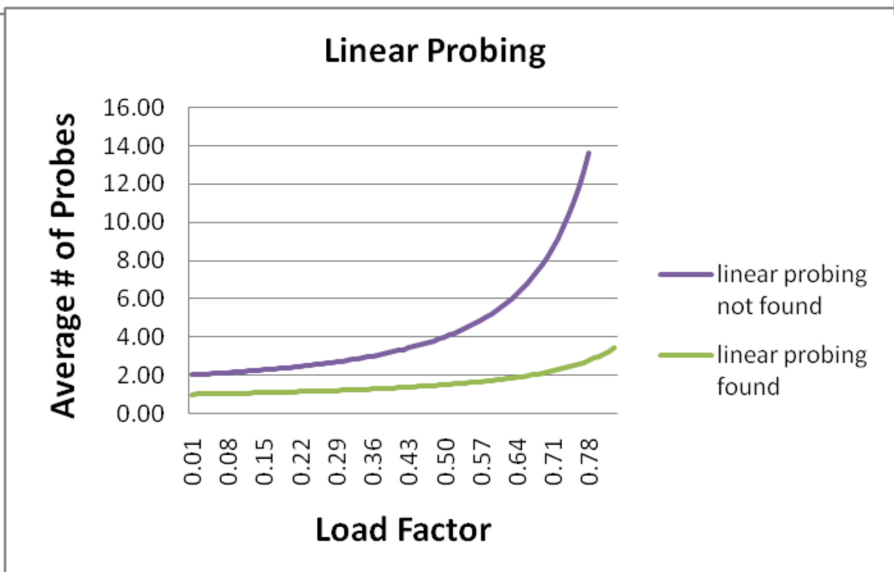
  Average # of probes given λ (in the limit as **`TableSize`** $\rightarrow \infty$)
  - Unsuccessful search (intuitive): $$\frac{1}{1-\lambda}$$
  - Successful search (less intuitive): $$\frac{1}{\lambda}\log_e\left(\frac{1}{1-\lambda}\right)$$

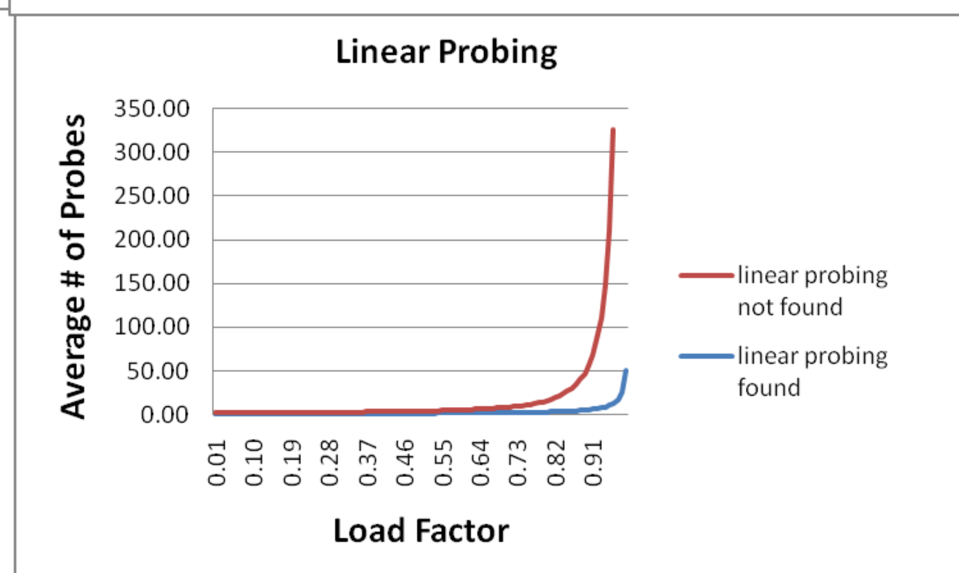
- Bottom line: unsuccessful bad (but not as bad as linear probing), but successful is not nearly as bad

Charts


Uniform Hashing
Average # of Probes
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
0.01
0.08
0.15
0.22
0.29
0.36
0.43
0.50
0.57
0.64
0.71
0.78
Load Factor
uniform hashing not found
uniform hashing found


Uniform Hashing
Average # of Probes
120.00
100.00
80.00
60.00
40.00
20.00
0.00
0.01
0.10
0.19
0.28
0.37
0.46
0.55
0.64
0.73
0.82
0.91
Load Factor
uniform hashing not found
uniform hashing found


Linear Probing
Average # of Probes
16.00
14.00
12.00
10.00
8.00
6.00
4.00
2.00
0.00
0.01
0.08
0.15
0.22
0.29
0.36
0.43
0.50
0.57
0.64
0.71
0.78
Load Factor
linear probing not found
linear probing found


Linear Probing
Average # of Probes
350.00
300.00
250.00
200.00
150.00
100.00
50.00
0.00
0.01
0.10
0.19
0.28
0.37
0.46
0.55
0.64
0.73
0.82
0.91
Load Factor
linear probing not found
linear probing found

Where are we?

- Separate Chaining is easy
  - **`find, insert, delete`** proportional to load factor on average if using unsorted linked list nodes
  - If using another data structure for buckets (e.g. AVL tree) , runtime is proportional to runtime for that structure.
- Open addressing uses probing, has clustering issues as table fills
  Why use it:
  - Less memory allocation?
    - Some run-time overhead for allocating linked list (or whatever) nodes; open addressing could be faster
  - Easier data representation?
- Now:
  - Growing the table when it gets too full (aka “rehashing”)
  - Relation between hashing/comparing and connection to Java

Rehashing

- As with array-based stacks/queues/lists, if table gets too full, create a bigger table and copy everything over

- With **separate chaining,** we get to decide what “too full” means
  – Keep load factor reasonable (e.g., < 1)?
  – Consider average or max size of non-empty chains?
- For **open addressing**, half-full is a good rule of thumb

- New table size
  – Twice-as-big is a good idea, except, uhm, that won’t be prime!
  – So go *about* twice-as-big
  – Can have a list of prime numbers in your code since you probably won’t grow more than 20-30 times, and then calculate after that

More on rehashing

- What if we copy all data to the same indices in the new table?
  - Will not work; we calculated the index based on **`TableSize`**

- Go through table, do standard insert for each into new table
  - Iterate over old table: O(n)
  - n inserts / calls to the hash function: $n \cdot O(1) = O(n)$

- Is there some way to avoid all those hash function calls?
  - Space/time tradeoff: Could store **`h(key)`** with each data item
  - Growing the table is still $O(n)$; saving **`h(key)`** only helps by a constant factor

Hashing and comparing

- Our use of int key can lead to us overlooking a critical detail:
  - We initially *hash* **`E`** to get a table index
  - While chaining or probing we need to determine if this is the **`E`** that I am looking for. Just need equality testing.

- So a hash table needs a hash function and a equality testing
  - In the Java library each object has an **`equals`** method and a **`hashCode`** method

```
class Object {
  boolean equals(Object o) {…}
  int hashCode() {…}
  …
}
```

Equal objects must hash the same

- The Java library (and your project hash table) make a very important assumption that clients must satisfy…
- Object-oriented way of saying it:

  If **`a.equals(b)`**, then we must require
  `a.hashCode()==b.hashCode()`

- Function object way of saying it:

  If **`c.compare(a,b) == 0`**, then we must require
  `h.hash(a) == h.hash(b)`

- If you ever override equals
  - You need to override hashCode also in a consistent way
  - See CoreJava book, Chapter 5 for other "gotchas" with equals

By the way: comparison has rules too

We have not emphasized important "rules" about comparison for:

- All our dictionaries
- Sorting (next major topic)

Comparison must impose a consistent, total ordering:

For all **`a`**, **`b`**, and **`c`**,

- If **`compare(a,b) < 0`**, then **`compare(b,a) > 0`**
- If **`compare(a,b) == 0`**, then **`compare(b,a) == 0`**
- If **`compare(a,b) < 0`** and **`compare(b,c) < 0`**, then **`compare(a,c) < 0`**

A Generally Good hashCode()


```
int result = 17; // start at a prime

foreach field f
   int fieldHashcode =
      boolean: (f ? 1: 0)
      byte, char, short, int: (int) f
      long: (int) (f ^ (f >>> 32))
      float: Float.floatToIntBits(f)
      double: Double.doubleToLongBits(f), then above
      Object: object.hashCode( )

      result = 31 * result + fieldHashcode;
return result;
```

Final word on hashing

- The hash table is one of the most important data structures
  - Efficient find, insert, and delete
  - Operations based on sorted order are not so efficient
  - Useful in many, many real-world applications
  - Popular topic for job interview questions
- Important to use a good hash function
  - Good distribution, Uses enough of key's values
  - Not overly expensive to calculate (bit shifts good!)
- Important to keep hash table at a good size
  - Prime #
  - Preferable λ depends on type of table
- What we skipped: Perfect hashing, universal hash functions, hopscotch hashing, cuckoo hashing
- Side-comment: hash functions have uses beyond hash tables
  - Examples: Cryptography, check-sums