CSE 332: Data Structures & Parallelism

Lecture 5: Algorithm Analysis II

Ruth Anderson

Autumn 2019

Today

- Analyzing Recursive Code
- Solving Recurrences

Analyzing code ("worst case")

Basic operations take "some amount of" constant time

- Arithmetic
- Assignment
- Access one Java field **or array index**
- Etc.

(This is an *approximation of reality*: a very useful "lie".)

Consecutive statements	Sum of time of each statement
Loops	Num iterations * time for loop body
Conditionals	Time of condition plus time of slower branch
Function Calls	Time of function's body
Recursion	Solve *recurrence equation*

Linear search

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   for(int i=0; i < arr.length; ++i)
      if(arr[i] == k)
        return true;
   return false;
}
```

Best case: 6 “ish” steps = $O(1)$

Worst case: 5 “ish” * (arr.length)
= $O(\text{arr.length})$

Analyzing Recursive Code

- Computing run-times gets interesting with recursion
- Say we want to perform some computation recursively on a list of size n
  - Conceptually, in each recursive call we:
    - Perform some amount of work, call it w(n)
    - Call the function recursively with a smaller portion of the list

- So, if we do w(n) work per step, and reduce the problem size in the next recursive call by 1, we do total work:

$$T(n)=w(n)+T(n-1)$$

- With some base case, like T(1)=5=O(1)

Example Recursive code: sum array

Recursive:

- Recurrence is some constant amount of work O(1) done *n* times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

Each time **`help`** is called, it does that O(1) amount of work, and then calls **`help`** again on a problem one less than previous problem size.

Recurrence Relation: $T(n) = O(1) + T(n\text{-}1)$

Solving Recurrence Relations

- Say we have the following recurrence relation:

  T(n)=6 “ish”+T(n-1)

  T(1)=9 “ish” ←base case

- Now we just need to solve it; that is, reduce it to a closed form.
- Start by writing it out:

  T(n)=6+T(n-1)

  =6+6+T(n-2)

  =6+6+6+T(n-3)

  =6+6+6+…+6+T(1) = 6+6+6+…+6+9

  =6k+T(n-k)

  =6k+9, where k is the # of times we expanded T()

- We expanded it out n-1 times, so

  T(n)=6k+T(n-k)

  =6(n-1)+T(1) = 6(n-1)+9

  =6n+3 = O(n)

Or When does n-k=1?
Answer: when k=n-1

Binary search

Best case:

Worst case:

2	**3**	**5**	**16**	**37**	**50**	**73**	**75**	**126**

Find an integer in a *sorted* array

- Can also be done non-recursively but "doesn't matter" here

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2; //i.e., lo+(hi-lo)/2
   if(lo==hi)        return false;
   if(arr[mid]==k) return true;
   if(arr[mid]< k) return help(arr,k,mid+1,hi);
   else            return help(arr,k,lo,mid);
}
```

Binary search

Best case: 9 "ish" steps = $O(1)$

Worst case: $T(n)$ = 10 "ish" + $T(n/2)$ where n is **`hi-lo`**

- $O($**`log`** $n)$ where n is **`array.length`**
- Solve *recurrence equation* to know that…

```
// requires array is sorted
// returns whether k is in array
boolean find(int[]arr, int k){
   return help(arr,k,0,arr.length);
}
boolean help(int[]arr, int k, int lo, int hi) {
   int mid = (hi+lo)/2;
   if(lo==hi)        return false;
   if(arr[mid]==k)   return true;
   if(arr[mid]< k)   return help(arr,k,mid+1,hi);
   else              return help(arr,k,lo,mid);
}
```

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   - $T(n) = 10 + T(n/2)$ $T(1) = 15$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.

3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case

Solving Recurrence Relations

1. Determine the recurrence relation. What is the base case?
   - $T(n) = 10 + T(n/2)$ $T(1) = 15$
2. "Expand" the original relation to find an equivalent general expression *in terms of the number of expansions*.
   - $T(n) = 10 + 10 + T(n/4)$
     $= 10 + 10 + 10 + T(n/8)$
     $= \ldots$
     $= 10k + T(n/(2^k))$ (where k is the number of expansions)
3. Find a closed-form expression by setting *the number of expansions* to a value which reduces the problem to a base case
   - $n/(2^k) = 1$ means $n = 2^k$ means $k = \log_2 n$
   - So $T(n) = 10 \log_2 n + 15$ (get to base case and do it)
   - So $T(n)$ is $O(\log n)$

sum array again

Two "obviously" linear algorithms: $T(n) = O(1) + T(n\text{-}1)$

Iterative:

```
int sum(int[] arr){
  int ans = 0;
  for(int i=0; i<arr.length; ++i)
     ans += arr[i];
  return ans;
}
```

Recursive:

- Recurrence is $c + c + \ldots + c$ for n times

```
int sum(int[] arr){
  return help(arr,0);
}
int help(int[]arr,int i) {
  if(i==arr.length)
    return 0;
  return arr[i] + help(arr,i+1);
}
```

What about a <u>binary</u> version of sum?

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

What about a binary version of sum?

```
int sum(int[] arr){
   return help(arr,0,arr.length);
}
int help(int[] arr, int lo, int hi) {
   if(lo==hi)   return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

Recurrence is $T(n) = O(1) + 2T(n/2)$

– 1 + 2 + 4 + 8 + … for **`log`** *n* times

– $2^{(\log n)} - 1$ which is proportional to *n* (by definition of logarithm)

Easier explanation: it adds each number once while doing little else

"Obvious": You can't do better than *O(n)* – have to read whole array

Parallelism teaser

- But suppose we could do two recursive calls *at the same time*
  - *Like having a friend do half the work for you!*

```
int sum(int[]arr){
   return help(arr,0,arr.length);
}
int help(int[]arr, int lo, int hi) {
   if(lo==hi)    return 0;
   if(lo==hi-1) return arr[lo];
   int mid = (hi+lo)/2;
   return help(arr,lo,mid) + help(arr,mid,hi);
}
```

- If you have as many "friends of friends" as needed, the recurrence is now $T(n) = O(1) + 1T(n/2)$
  - $O($**`log`** $n)$: same recurrence as for **`find`**

Really common recurrences

Should know how to solve recurrences but also recognize some really common ones:

$T(n) = O(1) + T(n/2)$	logarithmic	$O(\log n)$
$T(n) = O(1) + 2T(n/2)$	linear	$O(n)$
$T(n) = O(1) + T(n-1)$	linear	$O(n)$
$T(n) = O(n) + T(n-1)$	quadratic	$O(n^2)$
$T(n) = O(1) + 2T(n-1)$	exponential	$O(2^n)$
$T(n) = O(n) + T(n/2)$	linear	$O(n)$
$T(n) = O(n) + 2T(n/2)$	loglinear	$O(n \log n)$

Note big-Oh can also use more than one variable

- Example: can sum all elements of an *n*-by-*m* matrix in $O(nm)$