Announcements

• No class on Monday
Union Find Review

• Data: set of pairwise disjoint sets.
• Operations
  – Union – merge two sets to create their union
  – Find – determine which set an item appears in
• Amortized complexity
  – M Union and Find operations, on a set of size N
  – Runtime O(M log* N)
Spanning Tree in a Graph

- Connects all the vertices
- No cycles
Undirected Graph

- $G = (V,E)$
  - $V$ is a set of vertices (or nodes)
  - $E$ is a set of unordered pairs of vertices

$V = \{1,2,3,4,5,6,7\}$
$E = \{(1,2),(1,6),(1,5),(2,7),(2,3), (3,4),(4,7),(4,5),(5,6)\}$

2 and 3 are adjacent
2 is incident to edge (2,3)
Spanning Tree Problem

• Input: An undirected graph $G = (V,E)$. $G$ is connected.

• Output: $T \subseteq E$ such that
  – $(V,T)$ is a connected graph
  – $(V,T)$ has no cycles
Spanning Tree Algorithm

ST(Vertex i) {
    mark i;
    for each j adjacent to i {
        if (j is unmarked) {
            Add (i,j) to T;
            ST(j);
        }
    }
}

Main() {
    T = empty set;
    ST(1);
}
Finding a reliable routing subnetwork:

- edge cost = probability that it won’t fail
- Find the spanning tree that is least likely to fail
Example of a Spanning Tree

Probability of success = $0.85 \times 0.95 \times 0.89 \times 0.95 \times 1.0 \times 0.84$

$= 0.5735$
Minimum Spanning Trees

Given an undirected graph $G=(V,E)$, find a graph $G'=(V,E')$ such that:

- $E'$ is a subset of $E$
- $|E'| = |V| - 1$
- $G'$ is connected
- $\sum_{(u,v)\in E'} c_{uv}$ is minimal

$G'$ is a minimum spanning tree.
Minimum Spanning Tree Problem

• Input: Undirected Graph $G = (V,E)$ and $C(e)$ is the cost of edge $e$.

• Output: A spanning tree $T$ with minimum total cost. That is: $T$ that minimizes

$$C(T) = \sum_{e \in T} C(e)$$
Kruskal’s MST Algorithm

Idea: Grow a forest out of edges that do not create a cycle. Pick an edge with the smallest weight.

G=(V,E)
Kruskal’s Algorithm for MST

An *edge-based* greedy algorithm
Builds MST by greedily adding edges

1. Initialize with
   - empty MST
   - all vertices marked unconnected
   - all edges unmarked

2. While there are still unmarked edges
   a. Pick the lowest cost edge \((u, v)\) and mark it
   b. If \(u\) and \(v\) are not already connected, add \((u, v)\) to the MST and mark \(u\) and \(v\) as connected to each other

*Sound familiar?*
Example of for Kruskal

(7,4) (2,1) (7,5) (5,6) (5,4) (1,6) (2,7) (2,3) (3,4) (1,5)
0 1 1 2 2 3 3 3 3 4
Data Structures for Kruskal

- Sorted edge list

(7,4) (2,1) (7,5) (5,6) (5,4) (1,6) (2,7) (2,3) (3,4) (1,5)

<table>
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- Disjoint Union / Find
  - Union(a,b) - merge the disjoint sets named by a and b
  - Find(a) returns the name of the set containing a
Example of DU/F

(7,4) (2,1) (7,5) (5,6) (5,4) (1,6) (2,7) (2,3) (3,4) (1,5)

0  1  1  2  2  3  3  3  3  4
Kruskal’s Algorithm

- Add the cheapest edge that joins disjoint components
Kruskal’s Algorithm with DU / F

Sort the edges by increasing cost; Initialize A to be empty; for each edge (i,j) chosen in increasing order do 
  u := Find(i); 
  v := Find(j); 
  if not(u = v) then  
    add (i,j) to A; 
    Union(u,v);

This algorithm will work, but it goes through all the edges.

Is this always necessary?
void Graph::kruskal()
{
    int edgesAccepted = 0;
    DisjSet s(NUM_VERTICES);

    while (edgesAccepted < NUM_VERTICES - 1)
    {
        e = smallest weight edge not deleted yet;
        // edge e = (u, v)
       uset = s.find(u);
        vset = s.find(v);
        if (uset != vset)
        {
            edgesAccepted++;
            s.unionSets(uset, vset);
        }
    }
}

Total Cost:
Kruskal’s Algorithm: Correctness

It clearly generates a spanning tree. Call it $T_K$.

Suppose $T_K$ is not minimum:

Pick another spanning tree $T_{\text{min}}$ with lower cost than $T_K$

Pick the smallest edge $e_1 = (u, v)$ in $T_K$ that is not in $T_{\text{min}}$

$T_{\text{min}}$ already has a path $p$ in $T_{\text{min}}$ from $u$ to $v$

$\implies$ Adding $e_1$ to $T_{\text{min}}$ will create a cycle in $T_{\text{min}}$

Pick an edge $e_2$ in $p$ that Kruskal’s algorithm considered after adding $e_1$ (must exist: $u$ and $v$ unconnected when $e_1$ considered)

$\implies$ cost($e_2$) $\geq$ cost($e_1$)

$\implies$ can replace $e_2$ with $e_1$ in $T_{\text{min}}$ without increasing cost!

Keep doing this until $T_{\text{min}}$ is identical to $T_K$

$\implies$ $T_K$ must also be minimal – contradiction!
MST Application: Clustering

• Given a collection of points in an r-dimensional space, and an integer K, divide the points into K sets that are closest together.
Distance clustering

• Divide the data set into $K$ subsets to maximize the distance between any pair of sets

  $\text{dist} (S_1, S_2) = \min \{ \text{dist}(x, y) \mid x \in S_1, y \in S_2 \}$
Divide into 2 clusters
Divide into 3 clusters
Divide into 4 clusters
Distance Clustering Algorithm

Let $C = \{\{v_1\}, \{v_2\}, \ldots, \{v_n\}\}; \ T = \{ \}$

while $|C| > K$

Let $e = (u, v)$ with $u$ in $C_i$ and $v$ in $C_j$ be the minimum cost edge joining distinct sets in $C$

Replace $C_i$ and $C_j$ by $C_i \cup C_j$
K-clustering