0. Heaps
Insert 10, 7, 15, 17, 12, 20, 6, 32 into a min heap.
Now, insert the same values into a max heap.
Now, insert the same values into a min heap, but use Floyd’s buildHeap algorithm. **Solution:**

```
ninHeap
  6
 /   \
10   7
 /   /   \
17  12  20  15
   /   \
  32

naxHeap
  6
 /   \
32
 /   \
20  17
 /   /   \
15  12  10  6
   /   \
  7

floyd's
  6
 /   \
7   10
 /   /   \
17  12  20  15
   /   \
  32
```
1. Big-Oh Proofs

For each of the following, prove that \( f \in \mathcal{O}(g) \).

(a) \( f(n) = 7n \)
    \( g(n) = \frac{n}{10} \)

**Solution:** Choose \( c = 70 \), \( n_0 = 1 \). Then, note that \( 7n = \frac{70n}{10} \leq 70 \left( \frac{n}{10} \right) \) for all \( n \geq 1 \). So, \( f(n) \in \mathcal{O}(g(n)) \).

(b) \( f(n) = 1000 \)
    \( g(n) = 3n^3 \)

**Solution:** Choose \( c = 1 \), \( n_0 = 1000 \). Then, note that \( 1000 \leq n \leq 3n^3 \) for all \( n \geq 1000 \). So, \( f(n) \in \mathcal{O}(g(n)) \).

(c) \( f(n) = 7n^2 + 3n \)
    \( g(n) = n^4 \)

**Solution:** Choose \( c = 14 \), \( n_0 = 1 \). Then, note that \( 7n^2 + 3n \leq 7(n^4 + n^4) \leq 14n^4 \) for all \( n \geq 1 \). So, \( f(n) \in \mathcal{O}(g(n)) \).

(d) \( f(n) = n + 2n \lg n \)
    \( g(n) = n \lg n \)

**Solution:** Choose \( c = 3 \), \( n_0 = 2 \). Then, note that \( n + 2n \lg n \leq n \lg n + 2n \lg n = 3n \lg n \) for all \( n \geq 2 \). So, \( f(n) \in \mathcal{O}(g(n)) \).
2. Is Your Program Running? Better Catch It!

For each of the following, determine the asymptotic worst-case runtime in terms of \( n \).

(a)

```c
int x = 0;
for (int i = n; i >= 0; i--) {
    if ((i % 3) == 0) {
        break;
    }
    else {
        x += n;
    }
}
```

**Solution:** This is \( \Theta(1) \), because \( n \), \( n - 1 \), or \( n - 2 \) will be divisible by three. So, the loop runs at most 3 times.

(b)

```c
int x = 0;
for (int i = 0; i < n; i++) {
    for (int j = 0; j < (n * n / 3); j++) {
        x += j;
    }
}
```

**Solution:**

\[
\sum_{i=0}^{n-1} \sum_{j=0}^{n^2/3-1} 1 = \sum_{i=0}^{n-1} \frac{n^2}{3} = n \left( \frac{n^2}{3} \right) \in \Theta(n^3)
\]

(c)

```c
int x = 0;
for (int i = 0; i <= n; i++) {
    for (int j = 0; j < (i * i); j++) {
        x += j;
    }
}
```

**Solution:**

\[
\sum_{i=0}^{n} \sum_{j=0}^{i^2-1} 1 = \sum_{i=0}^{n} i^2 = \left( \frac{n(n+1)(2n+1)}{6} \right) \in \Theta(n^3)
\]
3. Induction Shminduction

Prove \( \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \) by induction on \( n \).

Solution:

Let \( P(n) \) be the statement \( \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \) for all \( n \in \mathbb{N} \). We prove \( P(n) \) by induction on \( n \).

Base Case. Note that \( \sum_{i=0}^{0} 2^i = 0 = 2^0 - 1 \). So, \( P(0) \) is true.

Induction Hypothesis. Suppose \( P(k) \) is true for some \( k \in \mathbb{N} \).

Induction Step. Note that

\[
\sum_{i=0}^{k+1} 2^i = \sum_{i=0}^{k} 2^i + 2^{k+1}
\]

\[
= 2^{k+1} - 1 + 2^{k+1} \quad \text{[By IH]}
\]

\[
= 2^{k+2} - 1
\]

Note that this is exactly \( P(k+1) \).

So, the claim is true by induction on \( n \).

4. The Implications of Asymptotics

For each of the following, determine if the statement is true or false.

(a) \( f(n) \in \Theta((g(n)) \rightarrow f(n) \in O(g(n)) \)

Solution:

This is true. By definition of \( f(n) \in \Theta((g(n)), \) we have \( f(n) \in O(g(n)) \).

(b) \( f(n) \in \Theta(g(n)) \rightarrow g(n) \in \Theta(f(n)) \)

Solution:

This is true. By definition of \( f(n) \in \Theta(g(n)), \) we have \( f(n) \in O(g(n)) \) and \( f(n) \in \Omega(g(n)) \). So, there exist \( n_0, n_1, c_0, c_1 > 0 \) such that \( f(n) \leq c_0 g(n) \) for all \( n \geq n_0 \) and \( f(n) \geq c_1 g(n) \) for all \( n \geq n_1 \). Define \( n_2 = \max(n_0, n_1) \) and note that both inequalities hold for all \( n \geq n_2 \). Then, dividing both sides by their constants, we have:

\[
g(n) \geq \frac{f(n)}{c_0}
\]

\[
g(n) \leq \frac{f(n)}{c_1}
\]

So, we’ve found constants \( \left( \frac{1}{c_0}, \frac{1}{c_1} \right) \) and a minimum \( n (n_2) \) that satisfy the definitions of Omega and Oh. It follows that \( g(n) \) is \( \Theta(f(n)) \).

(c) \( f(n) \in \Omega(g(n)) \rightarrow g(n) \in O(f(n)) \)
Solution:
This is true. This is basically identical to the previous part (except we only have to do half the work).

5. Asymptotic Analysis
For each of the following, determine if $f \in O(g)$, $f \in \Omega(g)$, $f \in \Theta(g)$, several of these, or none of these.

(a) $f(n) = \log n \quad g(n) = \log \log n$

Solution: $f(n) \in \Omega(g(n))$

(b) $f(n) = 2^n \quad g(n) = 3^n$

Solution: $f(n) \in O(g(n))$

(c) $f(n) = 2^{2n} \quad g(n) = 2^n$

Solution: $f(n) \in \Omega(g(n))$