Today

- Sorting
  - Comparison sorting
  - Beyond comparison sorting
The Big Picture

Simple algorithms: $O(n^2)$
- Insertion sort
- Selection sort
- Shell sort
  ...

Fancier algorithms: $O(n \log n)$
- Heap sort
- Merge sort
- Quick sort (avg)
  ...

Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: $O(n)$
- Bucket sort
- Radix sort

Handling huge data sets
- External sorting
How fast can we sort?

- Heapsort & mergesort have $O(n \log n)$ worst-case running time
- Quicksort has $O(n \log n)$ average-case running times
- These bounds are all tight, actually $\Theta(n \log n)$
- So maybe we need to dream up another algorithm with a lower asymptotic complexity, such as $O(n)$ or $O(n \log \log n)$
  - Instead: prove that this is impossible
    - *Assuming* our comparison *model*: The only operation an algorithm can perform on data items is a 2-element comparison
A Different View of Sorting

- Assume we have \( n \) elements to sort
  - And for simplicity, none are equal (no duplicates)

- How many *permutations* (possible orderings) of the elements?

- Example, \( n=3 \),
A Different View of Sorting

- Assume we have $n$ elements to sort
  - And for simplicity, none are equal (no duplicates)

- How many permutations (possible orderings) of the elements?

- Example, $n=3$, six possibilities
  
  \[
  \begin{align*}
  \end{align*}
  \]

- In general, $n$ choices for least element, then $n-1$ for next, then $n-2$ for next, …
  - $n(n-1)(n-2)\ldots(2)(1) = n!$ possible orderings
Describing every comparison sort

- A different way of thinking of sorting is that the sorting algorithm has to “find” the right answer among the $n!$ possible answers
  - Starts “knowing nothing”, “anything is possible”
  - Gains information with each comparison, eliminating some possibilities
    - Intuition: At best, each comparison can eliminate half of the remaining possibilities
  - In the end narrows down to a single possibility
Counting Comparisons

• Don’t know what the algorithm is, but it cannot make progress without doing comparisons
  – Eventually does a first comparison “is $a < b$?”
  – Can use the result to decide what second comparison to do
  – Etc.: comparison $k$ can be chosen based on first $k-1$ results

• What is the first comparison in:
  – Selection Sort?
  – Insertion Sort?
  – Quicksort?
  – Mergesort?
Counting Comparisons

- Don’t know what the algorithm is, but it cannot make progress without doing comparisons
  - Eventually does a first comparison “is \( a < b \)?”
  - Can use the result to decide what second comparison to do
  - Etc.: comparison \( k \) can be chosen based on first \( k-1 \) results

- Can represent this process as a decision tree
  - Nodes contain “set of remaining possibilities”
  - At root, anything is possible; no option eliminated
  - Edges are “answers from a comparison”
  - The algorithm does not actually build the tree; it’s what our proof uses to represent “the most the algorithm could know so far” as the algorithm progresses
One Decision Tree for n=3

- The leaves contain all the possible orderings of a, b, c
- A different algorithm would lead to a different tree
Example if \( a < c < b \)
What the decision tree tells us

- A binary tree because each comparison has 2 outcomes
  - Perform only comparisons between 2 elements: binary result
    - Ex: Is a<b? Yes or no?
  - We assume no duplicate elements
  - Assume algorithm doesn't ask redundant questions
- Because any data is possible, any algorithm needs to ask enough questions to produce all \( n! \) answers
  - Each answer is a different leaf
  - So the tree must be big enough to have \( n! \) leaves
  - Running any algorithm on any input will at best correspond to a root-to-leaf path in some decision tree with \( n! \) leaves
  - So no algorithm can have worst-case running time better than the height of a tree with \( n! \) leaves
    - Worst-case number-of-comparisons for an algorithm is an input leading to a longest path in algorithm's decision tree
Where are we

Proven: No comparison sort can have worst-case running time better than: the height of a binary tree with \( n! \) leaves
- Turns out average-case is same asymptotically
- A comparison sort could be worse than this height, but it cannot be better
- Fine, how tall is a binary tree with \( n! \) leaves?

Now: Show that a binary tree with \( n! \) leaves has height \( \Omega(n \log n) \)
- That is, \( n \log n \) is the lower bound, the height must be at least this, could be more, (in other words your comparison sorting algorithm could take longer than this, but it won’t be faster)
- Factorial function grows very quickly

Then we’ll conclude that: (Comparison) Sorting is \( \Omega(n \log n) \)
- This is an amazing computer-science result: proves all the clever programming in the world can’t sort in linear time!
Lower bound on Height

- A binary tree of height $h$ has at most how many leaves?
  \[ L \leq 2^h \]

- A binary tree with $L$ leaves has height at least:
  \[ h \geq \log_2 L \]

- The decision tree has how many leaves: $N!$
- So the decision tree has height:
  \[ h \geq \log(N!) \]
**Lower bound on Height**

- A binary tree of height $h$ has **at most how many** leaves?
  \[ L \leq 2^h \]

- A binary tree with $L$ leaves has height **at least**:
  \[ h \geq \log_2 L \]

- The decision tree has how many leaves: $N!$
- So the decision tree has height:
  \[ h \geq \log_2 N! \]
Lower bound on height

- The height of a binary tree with \( L \) leaves is at least \( \log_2 L \).
- So the height of our decision tree, \( h \):
  \[
  h \geq \log_2 (n!)
  \]
  property of binary trees
  \[
  = \log_2 (n^*(n-1)^*(n-2)\ldots(2)(1))
  \]
  definition of factorial
  \[
  = \log_2 n + \log_2 (n-1) + \ldots + \log_2 1
  \]
  property of logarithms
  \[
  \geq \log_2 n + \log_2 (n-1) + \ldots + \log_2 (n/2)
  \]
  keep first \( n/2 \) terms
  \[
  \geq (n/2) \log_2 (n/2)
  \]
  each of the \( n/2 \) terms left is \( \geq \log_2 (n/2) \)
  \[
  = (n/2)(\log_2 n - \log_2 2)
  \]
  property of logarithms
  \[
  = (1/2)n \log_2 n - (1/2)n
  \]
  arithmetic
  \[
  = \Omega (n \log n)
  \]
The Big Picture

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- ...

Fancier algorithms: $O(n \log n)$
- Heap sort
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Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: $O(n)$
- Bucket sort
- Radix sort

Handling huge datasets
- External sorting

How???
- Change the model – assume more than 'compare(a,b)'

10/31/2016
**BucketSort (a.k.a. BinSort)**

- If all values to be sorted are known to be integers between 1 and $K$ (or any small range),
  - Create an array of size $K$, and put each element in its proper bucket (a.k.a. bin)
  - If data is only integers, no need to store more than a *count* of how many times that bucket has been used
- Output result via linear pass through array of buckets

<table>
<thead>
<tr>
<th>count array</th>
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</table>

- Example:
  - $K=5$
  - Input: $(5, 1, 3, 4, 3, 2, 1, 1, 5, 4, 5)$
  - Output: $(1, 1, 2, 3, 3, 4, 4, 5, 5, 5)$

$O(N)$

$O(K+N)$
**BucketSort (a.k.a. BinSort)**

- If all values to be sorted are known to be integers between 1 and $K$ (or any small range),
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</table>

- Example:
  - $K=5$
  - input $(5,1,3,4,3,2,1,1,5,4,5)$
  - output: $1,1,1,2,3,3,4,4,5,5,5$

What is the running time?
Analyzing bucket sort

- Overall: $O(n+K)$
  - Linear in $n$, but also linear in $K$
  - $\Omega(n \log n)$ lower bound does not apply because this is not a comparison sort

- Good when range, $K$, is smaller (or not much larger) than $n$
  - (We don’t spend time doing lots of comparisons of duplicates!)

- Bad when $K$ is much larger than $n$
  - Wasted space; wasted time during final linear $O(K)$ pass

- For data in addition to integer keys, use list at each bucket
Bucket Sort with Data

- Most real lists aren't just #'s; we have data
- Each bucket is a list (say, linked list)
- To add to a bucket, place at end O(1) (keep pointer to last element)

<table>
<thead>
<tr>
<th>count array</th>
<th>Rocky V</th>
<th>Harry Potter</th>
<th>Casablanca</th>
<th>Star Wars</th>
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Bucket sort illustrates a more general trick:
How might you implement a heap for a small range of integer priorities in a similar manner...

Example: Movie ratings:
1=bad,... 5=excellent

Input=
5: Casablanca
3: Harry Potter movies
1: Rocky V
5: Star Wars

Result: 1: Rocky V, 3: Harry Potter, 5: Casablanca, 5: Star Wars
This result is stable; Casablanca still before Star Wars
Radix sort

- Radix = “the base of a number system”
  - Examples will use 10 because we are used to that
  - In implementations use larger numbers
    - For example, for ASCII strings, might use 128
- Idea:
  - Bucket sort on one digit at a time
    - Number of buckets = radix
    - Starting with least significant digit, sort with Bucket Sort
    - Keeping sort stable
  - Do one pass per digit
- Invariant: After $k$ passes, the last $k$ digits are sorted

Aside: Origins go back to the 1890 U.S. census
Example

Radix = 10

<table>
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<tr>
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Input: 478
      537
      9
      721
      3
      143
      67

First pass:
1. bucket sort by ones digit
2. Iterate thru and collect into a list
   • List is sorted by first digit

Order now: 721
      3
      143
      478
      38
      9

10/31/2016
### Example

Radix = 10

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Order was: 721, 3, 143, 537, 67, 478, 38, 9

Second pass: stable bucket sort by tens digit

If we chop off the 100's place, these #s are sorted

Order now: 3, 9, 721, 537, 38, 143, 67, 478, 38, 9

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Example

Radix = 10

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Order was:

- 3
- 9
- 721
- 537
- 143
- 38
- 67

Order now:

- 3
- 9
- 38
- 67
- 143
- 478
- 537
- 721
- 67
- 478

Third pass:

stable bucket sort by 100s digit

Only 3 digits: We’re done!

10/31/2016
Student Activity

**RadixSort**

- Input: 126, 328, 636, 341, 416, 131, 328

**BucketSort on lsd:**

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**BucketSort on next-higher digit:**

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**BucketSort on msd:**

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Analysis of Radix Sort

Performance depends on:

- Input size: \( n \)
- Number of buckets = Radix: \( B \)
  - e.g. Base 10 #: 10; binary #: 2; Alpha-numeric char: 62
- Number of passes = “Digits”: \( P \)
  - e.g. Ages of people: 3; Phone #: 10; Person’s name: ?

- Work per pass is 1 bucket sort: __________
  - Each pass is a Bucket Sort
- Total work is __________
  - We do ‘P’ passes, each of which is a Bucket Sort
Analysis of Radix Sort

Performance depends on:

• Input size: \( n \)
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  – e.g. Base 10 #: 10; binary #: 2; Alpha-numeric char: 62
• Number of passes = “Digits”: \( P \)
  – e.g. Ages of people: 3; Phone #: 10; Person’s name: ?

• Work per pass is 1 bucket sort: \( O(B+n) \)
  – Each pass is a Bucket Sort
• Total work is \( O(P(B+n)) \)
  – We do ‘P’ passes, each of which is a Bucket Sort
Comparison to Comparison Sorts

Compared to comparison sorts, sometimes a win, but often not
  – Example: Strings of English letters up to length 15
    • Approximate run-time: $15^* (52 + n)$
    • This is less than $n \log n$ only if $n > 33,000$
    • Of course, cross-over point depends on constant factors of the implementations plus $P$ and $B$
      – And radix sort can have poor locality properties
  – Not really practical for many classes of keys
    • Strings: Lots of buckets
Recap: Features of Sorting Algorithms

In-place

- Sorted items occupy the same space as the original items. (No copying required, only $O(1)$ extra space if any.)

Stable

- Items in input with the same value end up in the same order as when they began.

Examples:

- Merge Sort - not in place, stable
- Quick Sort - in place, not stable
Sorting massive data: External Sorting

Need sorting algorithms that **minimize disk/tape access** time:

- Quicksort and Heapsort both jump all over the array, leading to expensive random disk accesses
- Mergesort scans linearly through arrays, leading to (relatively) efficient sequential disk access

Basic Idea:

- Load chunk of data into Memory, sort, store this “run” on disk/tape
- Use the Merge routine from Mergesort to merge runs
- Repeat until you have only one run (one sorted chunk)

- Mergesort can leverage multiple disks
- Weiss gives some examples
Sorting Summary

- Simple $O(n^2)$ sorts can be fastest for small $n$
  - selection sort, insertion sort (latter linear for mostly-sorted)
  - good for “below a cut-off” to help divide-and-conquer sorts
- $O(n \log n)$ sorts
  - heap sort, in-place but not stable nor parallelizable
  - merge sort, not in-place but stable and works as external sort
  - quick sort, in-place but not stable and $O(n^2)$ in worst-case
    - often fastest, but depends on costs of comparisons/copies
- $\Omega(n \log n)$ is worst-case and average lower-bound for sorting by comparisons
- Non-comparison sorts
  - Bucket sort good for small number of key values
  - Radix sort uses fewer buckets and more phases
- Best way to sort? It depends!