CSE 332: Data Structures & Parallelism

Lecture 12: Comparison Sorting

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Today

- Dictionaries
  - Hashing
- Sorting
  - Comparison sorting
Introduction to sorting

- Stacks, queues, priority queues, and dictionaries all focused on providing one element at a time
- But often we know we want “all the data items” in some order
  - Anyone can sort, but a computer can sort faster
  - Very common to need data sorted somehow
    - Alphabetical list of people
    - Population list of countries
    - Search engine results by relevance
    - ...
- Different algorithms have different asymptotic and constant-factor trade-offs
  - No single ‘best’ sort for all scenarios
  - Knowing one way to sort just isn’t enough
More reasons to sort

General technique in computing:
Preprocess (e.g. sort) data to make subsequent operations faster

Example: Sort the data so that you can
- Find the $k^{th}$ largest in constant time for any $k$
- Perform binary search to find an element in logarithmic time

Whether the benefit of the preprocessing depends on
- How often the data will change
- How much data there is
The main problem, stated carefully

For now we will assume we have \( n \) comparable elements in an array and we want to rearrange them to be in increasing order

Input:
- An array \( A \) of data records
- A key value in each data record
- A comparison function (consistent and total)
  - Given keys \( a \) & \( b \), what is their relative ordering? \( <, =, > \)?
  - Ex: keys that implement Comparable or have a Comparator that can handle them

Effect:
- Reorganize the elements of \( A \) such that for any \( i \) and \( j \), if \( i < j \) then \( A[i] \leq A[j] \)
- Usually unspoken assumption: \( A \) must have all the same data it started with
- Could also sort in reverse order, of course

An algorithm doing this is a comparison sort
Variations on the basic problem

1. Maybe elements are in a linked list (could convert to array and back in linear time, but some algorithms needn’t do so)
2. Maybe in the case of ties we should preserve the original ordering
   – Sorts that do this naturally are called stable sorts
   – One way to sort twice, Ex: Sort movies by year, then for ties, alphabetically
3. Maybe we must not use more than $O(1)$ “auxiliary space”
   – Sorts meeting this requirement are called ‘in-place’ sorts
   – Not allowed to allocate extra array (at least not with size $O(n)$), but can allocate $O(1)$ # of variables
   – All work done by swapping around in the array
4. Maybe we can do more with elements than just compare
   – Comparison sorts assume we work using a binary ‘compare’ operator
   – In special cases we can sometimes get faster algorithms
5. Maybe we have too much data to fit in memory
   – Use an “external sorting” algorithm
Sorting: The Big Picture

Simple algorithms: $O(n^2)$
- Insertion sort
- Selection sort
- Shell sort
- ...

Fancier algorithms: $O(n \log n)$
- Heap sort
- Merge sort
- Quick sort (avg)
- ...

Comparison lower bound: $\Omega(n \log n)$

Specialized algorithms: $O(n)$
- Bucket sort
- Radix sort

Handling huge data sets
- External sorting
Insertion Sort - like inserting cards into your hand

• Idea: At step \( k \), put the \( k \)-th element in the correct position among the first \( k \) elements

• Alternate way of saying this:
  – Sort first two elements
  – Now insert 3\(^{rd}\) element in order
  – Now insert 4\(^{th}\) element in order
  – ...

• “Loop invariant”: when loop index is \( i \), first \( i \) elements are sorted

• Time?
  Best-case \( \mathcal{O}(n) \)  Worst-case \( \mathcal{O}(n^2) \) “Average” case \( \mathcal{O}(n^2) \)

Already Sorted  Reverse Sorted Order

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**Insertion Sort**

- Idea: At step $k$, put the $k^{th}$ element in the correct position among the first $k$ elements

- Alternate way of saying this:
  - Sort first two elements
  - Now insert 3rd element in order
  - Now insert 4th element in order
  - ...

- “Loop invariant”: when loop index is $i$, first $i$ elements are sorted

- Time?
  
<table>
<thead>
<tr>
<th>Best-case</th>
<th>Worst-case</th>
<th>Average case</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(n)$</td>
<td>$O(n^2)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>start sorted</td>
<td>start reverse sorted</td>
<td>(see text)</td>
</tr>
</tbody>
</table>

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Selection sort

• Idea: At step $k$, find the smallest element among the not-yet-sorted elements and put it at position $k$

• Alternate way of saying this:
  – Find smallest element, put it 1st
  – Find next smallest element, put it 2nd
  – Find next smallest element, put it 3rd
  – ...

• “Loop invariant”: when loop index is $i$, first $i$ elements are the $i$ smallest elements in sorted order

• Time?
  Best-case $O(n^2)$  Worst-case $O(n^2)$  "Average" case $O(n^2)$
Selection sort

• Idea: At step $k$, find the smallest element among the not-yet-sorted elements and put it at position $k$

• Alternate way of saying this:
  – Find smallest element, put it $1^{\text{st}}$
  – Find next smallest element, put it $2^{\text{nd}}$
  – Find next smallest element, put it $3^{\text{rd}}$
  – ...

• “Loop invariant”: when loop index is $i$, first $i$ elements are the $i$ smallest elements in sorted order

• Time?
  
  Best-case $O(n^2)$  Worst-case $O(n^2)$  “Average” case $O(n^2)$
  
  Always $T(1) = 1$ and $T(n) = \underline{n} + T(n-1)$
Insertion Sort vs. Selection Sort

- Different algorithms

- Solve the same problem

- Have the same worst-case and average-case asymptotic complexity
  - Insertion-sort has better best-case complexity; preferable when input is “mostly sorted”

- Other algorithms are more efficient for non-small arrays that are not already almost sorted
  - Insertion sort may do well on small arrays
Aside: We won’t cover Bubble Sort

• It doesn’t have good asymptotic complexity: $O(n^2)$

• It’s not particularly efficient with respect to common factors

• Basically, almost everything it is good at, some other algorithm is at least as good at

• Some people seem to teach it just because someone taught it to them

• For fun see: "Bubble Sort: An Archaeological Algorithmic Analysis", Owen Astrachan, SIGCSE 2003

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Sorting: The Big Picture

**Simple algorithms:** $O(n^2)$
- Insertion sort
- Selection sort
- Shell sort
  ...

**Fancier algorithms:** $O(n \log n)$
- Heap sort
- Merge sort
- Quick sort (avg)
  ...

**Comparison lower bound:** $\Omega(n \log n)$

**Specialized algorithms:** $O(n)$
- Bucket sort
- Radix sort

**Handling huge data sets**
- External sorting
Heap sort

• Sorting with a heap is easy:
  - insert each arr[i], better yet use buildHeap
  - for (i=0; i < arr.length; i++)
    arr[i] = deleteMin();

• Worst-case running time:
  \[ O(n) \] \( \rightarrow \) \[ O(n \log n) \]

• We have the array-to-sort and the heap
  - So this is not an in-place sort
  - There’s a trick to make it in-place...
Heap sort

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  - insert each arr[i], better yet use buildHeap
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• Worst-case running time: $O(n \log n)$ why?

• We have the array-to-sort and the heap
  – So this is not an in-place sort
  – There’s a trick to make it in-place...
**In-place heap sort**

- Treat the initial array as a heap (via `buildHeap`)
- When you delete the $i$th element, put it at `arr[n-i]`
  - It's not part of the heap anymore!
“AVL sort”

- How?
“AVL sort”

• We can also use a balanced tree to:
  – insert each element: total time $O(n \log n)$
  – Do an in-order traversal $O(n)$

• But this cannot be made in-place and has worse constant factors than heap sort
  – both are $O(n \log n)$ in worst, best, and average case
  – neither parallelizes well
  – heap sort is better

• Don’t even think about trying to sort with a hash table...
Divide and conquer

Very important technique in algorithm design

1. Divide problem into smaller parts

2. Solve the parts independently
   - Think recursion
   - Or potential parallelism

3. Combine solution of parts to produce overall solution

Ex: Sort each half of the array, combine together; to sort each half, split into halves...
**Divide-and-conquer sorting**

Two great sorting methods are fundamentally divide-and-conquer:

1. **Mergesort:**
   - Sort the left half of the elements (recursively)
   - Sort the right half of the elements (recursively)
   - Merge the two sorted halves into a sorted whole

2. **Quicksort:**
   - Pick a “pivot” element
   - Divide elements into those less-than pivot and those greater-than pivot
   - Sort the two divisions (recursively on each)
   - Answer is \([\text{sorted-less-than then pivot then sorted-greater-than}]\)
Mergesort

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
<td>9</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>

- To sort array from position \(lo\) to position \(hi\):
  - If range is 1 element long, it's sorted! (Base case)
  - Else, split into two halves:
    - Sort from \(lo\) to \((hi+lo)/2\)
    - Sort from \((hi+lo)/2\) to \(hi\)
    - Merge the two halves together

- Merging takes two sorted parts and sorts everything
  - \(O(n)\) but requires auxiliary space…

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Example, focus on merging

Start with:

\[ \begin{array}{cccccccc}
8 & 2 & 9 & 4 & 5 & 3 & 1 & 6 \\
\end{array} \]

After we return from left and right recursive calls (pretend it works for now):

\[ \begin{array}{cccccccc}
2 & 4 & 8 & 9 & 1 & 3 & 5 & 6 \\
\end{array} \]

Merge:

Use 3 “fingers” aux and 1 more array

(After merge, copy back to original array)
Example, focus on merging

Start with:

```
8 2 9 4 5 3 1 6
```

After recursion:
(not magic 😊)

```
2 4 8 9 1 3 5 6
```

Merge:
Use 3 “fingers”
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Example, focus on merging

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| 8 | 2 | 9 | 4 | 5 | 3 | 1 | 6 |

After recursion:
(not magic 😊)

| 2 | 4 | 8 | 9 | 1 | 3 | 5 | 6 |

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After recursion: 2 4 8 9 1 3 5 6 (not magic 😊)

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Example, focus on merging

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```
  8 2 9 4 5 3 1 6
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After recursion:
(not magic 😊)

```
  2 4 8 9 1 3 5 6
```

Merge:
Use 3 “fingers”
and 1 more array

```
  1 2 3 4 5 6 8 9
```

(After merge, copy back to original array)

```
  1 2 3 4 5 6 8 9
```
Mergesort example: Recursively splitting list in half

```
| 8 | 2 | 9 | 4 | 5 | 3 | 1 | 6 |
```

Divide

```
| 8 | 2 |
| 9 | 4 |
```

Divide

```
| 5 | 3 |
| 1 | 6 |
```

Divide

```
| 8 | 2 |
| 9 | 4 |
```

Divide

```
| 5 | 3 |
| 1 | 6 |
```

Divide

```
| 8 |
| 2 |
```

Divide

```
| 9 |
| 4 |
```

Divide

```
| 5 |
| 3 |
```

Divide

```
| 1 |
| 6 |
```

Divide

```
1 element
```

```
| 8 |
| 2 |
| 9 |
| 4 |
| 5 |
| 3 |
| 1 |
| 6 |
```

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Mergesort example: Merge as we return from recursive calls

8 2 9 4 5 3 1 6

Divide

Divide

Divide

Divide

1 element

Merge

Merge

Merge

When a recursive call ends, it’s sub-arrays are each in order; just need to merge them in order together
Mergesort example: Merge as we return from recursive calls

We need another array in which to do each merging step; merge results into there, then copy back to original array.
Mergesort, some details: saving a little time

- What if the final steps of our merging looked like the following:

```
  Main array

  2 4 5 6 1 3 8 9
```

```
  Auxiliary array

  1 2 3 4 5 6
```

- Seems kind of wasteful to copy 8 & 9 to the auxiliary array just to copy them immediately back...
Mergesort, some details: saving a little time

- Unnecessary to copy 'dregs' over to auxiliary array
  - If left-side finishes first, just stop the merge & copy the auxiliary array:

- If right-side finishes first, copy dregs directly into right side, then copy auxiliary array
Some details: saving space / copying

Simplest / worst approach:
  Use a new auxiliary array of size \((hi-1o)\) for every merge
  Returning from a recursive call? Allocate a new array!

Better:
  Reuse same auxiliary array of size \(n\) for every merging stage
  Allocate auxiliary array at beginning, use throughout

Best (but a little tricky):
  Don’t copy back – at 2\textsuperscript{nd}, 4\textsuperscript{th}, 6\textsuperscript{th}, … merging stages, use the original array as the auxiliary array and vice-versa
  – Need one copy at end if number of stages is odd
Picture of the “best” from previous slide:
Allocate one auxiliary array, switch each step

First recurse down to lists of size 1
As we return from the recursion, switch off arrays

Arguably easier to code up without recursion at all

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Linked lists and big data

We defined the sorting problem as over an array, but sometimes you want to sort linked lists.

One approach:
- Convert to array: $O(n)$
- Sort: $O(n \log n)$
- Convert back to list: $O(n)$

Or: mergesort works very nicely on linked lists directly
- heapsort and quicksort do not
- insertion sort and selection sort do but they’re slower

Mergesort is also the sort of choice for external sorting
- Linear merges minimize disk accesses
Mergesort Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time (and space):

To sort \( n \) elements, we:
- Return immediately if \( n=1 \)
- Else do 2 subproblems of size \( n/2 \) and then an \( O(n) \) merge

Recurrence relation?
Mergesort Analysis

Having defined an algorithm and argued it is correct, we should analyze its running time (and space):

To sort \( n \) elements, we:

- Return immediately if \( n=1 \)
- Else do 2 subproblems of size \( n/2 \) and then an \( O(n) \) merge

Recurrence relation:

\[
T(1) = c_1 \\
T(n) = 2T(n/2) + c_2n
\]
MergeSort Recurrence

(For simplicity let constants be 1 – no effect on asymptotic answer)

\[
\begin{align*}
T(1) &= 1 \\
T(n) &= 2T(n/2) + n \\
&= 2(2T(n/4) + n/2) + n \\
&= 4T(n/4) + 2n \\
&= 4(2T(n/8) + n/4) + 2n \\
&= 8T(n/8) + 3n \\
&\vdots \text{(after k expansions)} \\
&= 2^kT(n/2^k) + kn
\end{align*}
\]

So total is \(2^kT(n/2^k) + kn\) where \(n/2^k = 1\), i.e., \(\log n = k\)

That is, \(2^{\log n} T(1) + n \log n\) = \(n + n \log n\) = \(O(n \log n)\)
Or more intuitively...

This recurrence comes up often enough you should just “know” it’s $O(n \log n)$.

Merge sort is relatively easy to intuit (best, worst, and average):
- The recursion “tree” will have $\log n$ height
- At each level we do a total amount of merging equal to $n$
Quicksort

• Also uses divide-and-conquer
  – Recursively chop into halves
  – But, instead of doing all the work as we merge together, we’ll
do all the work as we recursively split into halves
  – Also unlike MergeSort, does not need auxiliary space

• $O(n \log n)$ on average ☺, but $O(n^2)$ worst-case ☹
  – MergeSort is always $O(n \log n)$
  – So why use QuickSort?

• Can be faster than mergesort
  – Often believed to be faster
  – Quicksort does fewer copies and more comparisons, so it
depends on the relative cost of these two operations!
Quicksort Overview

1. Pick a pivot element
   - Hopefully an element ~median
   - Good QuickSort performance depends on good choice of pivot; we’ll see why later, and talk about good pivot selection later

2. Partition all the data into:
   A. The elements less than the pivot
   B. The pivot
   C. The elements greater than the pivot

3. Recursively sort A and C
4. The answer is, “as simple as A, B, C”

(Alas, there are some details lurking in this algorithm)
Quicksort: Think in terms of sets

S

select pivot value

S_1

partition S

S_2

QuickSort(S_1) and QuickSort(S_2)

Presto! S is sorted

[Weiss]

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Quicksort Example, showing recursion

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Quicksort Details

We have not yet explained:

- How to pick the pivot element
  - Any choice is correct: data will end up sorted
  - But as analysis will show, want the two partitions to be about equal in size

- How to implement partitioning
  - In linear time
  - In place
**Pivots**

- **Best pivot?**
  - Median
  - Halve each time

- **Worst pivot?**
  - Greatest/least element
  - Reduce to problem of size 1 smaller
  - $O(n^2)$
QuickSort: Potential pivot rules

While sorting $arr$ from $lo$ (inclusive) to $hi$ (exclusive)...

- Pick $arr[lo]$ or $arr[hi-1]$
  - Fast, but worst-case is (mostly) sorted input

- Pick random element in the range
  - Does as well as any technique, but (pseudo)random number generation can be slow
  - (Still probably the most elegant approach)

- Median of 3, e.g., $arr[lo]$, $arr[hi-1]$, $arr[(hi+lo)/2]$
  - Common heuristic that tends to work well
Partitioning

• That is, given 8, 4, 2, 9, 3, 5, 7 and pivot 5
  – Dividing into left half & right half (based on pivot)

• Conceptually simple, but hardest part to code up correctly
  – After picking pivot, need to partition
    • Ideally in linear time
    • Ideally in place

• Ideas?
Partitioning

- One approach (there are slightly fancier ones):
  1. Swap pivot with arr[lo]; move it ‘out of the way’
  2. Use two fingers i and j, starting at lo+1 and hi-1 (start & end of range, apart from pivot)
  3. Move from right until we hit something less than the pivot; belongs on left side
     Move from left until we hit something greater than the pivot; belongs on right side
     Swap these two; keep moving inward
     while (i < j)
       if (arr[j] > pivot) j--
       else if (arr[i] < pivot) i++
       else swap arr[i] with arr[j]
  4. Put pivot back in middle (Swap with arr[i])
Quicksort Example

- Step one: pick pivot as median of 3
  - $l_o = 0$, $h_i = 10$

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
8 & 1 & 4 & 9 & 0 & 3 & 5 & 2 & 7 & 6
\end{array}
\]

- Step two: move pivot to the $l_o$ position

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
6 & 1 & 4 & 9 & 0 & 3 & 5 & 2 & 7 & 8
\end{array}
\]
QuickSort Example

Often have more than one swap during partition – this is a short example

Now partition in place

Move fingers

Swap

Move fingers

Move pivot

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Quicksort Analysis

- Best-case?

- Worst-case?

- Average-case?
Quicksort Analysis

- Best-case: Pivot is always the median
  \[ T(0) = T(1) = 1 \]
  \[ T(n) = 2T(n/2) + n \] -- linear-time partition
  Same recurrence as mergesort: \( O(n \log n) \)

- Worst-case: Pivot is always smallest or largest element
  \[ T(0) = T(1) = 1 \]
  \[ T(n) = T(n-1) + n \]
  Basically same recurrence as selection sort: \( O(n^2) \)

- Average-case (e.g., with random pivot)
  \( O(n \log n) \), not responsible for proof (in text)
Quicksort Cutoffs

- For small $n$, all that recursion tends to cost more than doing a quadratic sort
  - Remember asymptotic complexity is for large $n$
  - Also, recursive calls add a lot of overhead for small $n$
- Common engineering technique: switch to a different algorithm for subproblems below a cutoff
  - Reasonable rule of thumb: use insertion sort for $n < 10$
- Notes:
  - Could also use a cutoff for merge sort
  - Cutoffs are also the norm with parallel algorithms
    - switch to sequential algorithm
  - None of this affects asymptotic complexity
Quicksort Cutoff skeleton

```c
void quicksort(int[] arr, int lo, int hi) {
    if (hi - lo < CUTOFF)
        insertionSort(arr, lo, hi);
    else
        ...
}
```

Notice how this cuts out the vast majority of the recursive calls
- Think of the recursive calls to quicksort as a tree
- Trims out the bottom layers of the tree

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