CSE 332: Data Abstractions

Lecture 15: Topological Sort / Graph Traversals

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Today

- Graphs
  - Representations
  - Topological Sort
  - Graph Traversals
Topological Sort

Problem: Given a DAG $G = (V, E)$, output all the vertices in order such that if no vertex appears before any other vertex that has an edge to it.

Example input:

Example output:

142, 126, 143, 311, 331, 332, 312, 341, 351, 333, 440, 352
Valid Topological Sorts:
Questions and comments

• Why do we perform topological sorts only on DAGs?

• Is there always a unique answer?

• What DAGs have exactly 1 answer?

• Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it.
Questions and comments

• Why do we perform topological sorts only on DAGs?
  – Because a cycle means there is no correct answer

• Is there always a unique answer?
  – No, there can be 1 or more answers; depends on the graph

• What DAGs have exactly 1 answer?
  – Lists

• Terminology: A DAG represents a partial order and a topological sort produces a total order that is consistent with it
Topological Sort Uses

- Figuring out how to finish your degree
- Computing the order in which to recompute cells in a spreadsheet
- Determining the order to compile files using a Makefile
- In general, taking a dependency graph and coming up with an order of execution
A First Algorithm for Topological Sort

1. Label (“mark”) each vertex with its in-degree
   – Think “write in a field in the vertex”
   – Could also do this via a data structure (e.g., array) on the side

2. While there are vertices not yet output:
   a) Choose a vertex \( v \) with labeled with in-degree of 0
   b) Output \( v \) and conceptually remove it from the graph
   c) For each vertex \( u \) adjacent to \( v \) (i.e. \( u \) such that \( (v,u) \) in \( E \)),
      decrement the in-degree of \( u \)
Example

Node: 126 142 143 311 312 331 332 333 341 351 352 440

Removed?

In-degree: 0 0 2 1 2 1 1 2 1 1 1 1
Example

Output: 126

Node: 126 142 143 311 312 331 332 333 341 351 352 440
Removed? x
In-degree: 0 0 2 1 2 1 1 2 1 1 1 1 1 1
Example

Node: 126 142 143 311 312 331 332 333 341 351 352 440
Removed? x x
In-degree: 0 0 2 1 2 1 1 2 1 1 1 1

Output: 126 142

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Example

Node: 126 142 143 311 312 331 332 333 341 351 352 440
Removed?  x  x  x
In-degree: 0 0 2 1 2 1 1 2 1 1 1 1
           1 0 0 0 0 0 0
           0

Output: 126
        142
        143
Example

Node: 126 142 143 311 312 331 332 333 341 351 352 440

Removed?: x x x x

In-degree: 0 0 2 1 2 1 1 2 1 1 1 1

Output: 126 142 143 311
Example

Node: 126 142 143 311 312 331 332 333 341 351 352 440
Removed? x x x x x x
In-degree: 0 0 2 1 2 1 1 2 1 1 1 1

Output: 126
142
143
311
331
Example

Node:  126 142 143  311  312  331  332  333  341  351  352  440

Removed?   x   x   x   x   x x x

In-degree:  0 0 2 1 2 1 1 1 1 1

Output:  126
         142
         143
         311
         331
         332

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Example

Node: 126 142 143 311 312 331 332 333 341 351 352 440
Removed? x x x x x x x x
In-degree: 0 0 2 1 1 2 1 1 2 1 1 1
...

Output: 126
142
143
311
331
332
312
Example

Node: 126 142 143 311 312 331 332 333 341 351 352 440
Removed? x x x x x x x x x x
In-degree: 0 0 2 1 2 1 1 2 1 1 1 1
          1 0 1 0 0 1 0 0 0 0 0 0
          0 0

Output: 126 142 143 311 331 332 333 341 351 352 440...
Example

Node:  126 142 143 311 312 331 332 333 341 351 352 440
Removed?  x  x  x  x  x  x  x  x  x  x  x  x
In-degree:  0  0  2  1  2  1  1  1  1  1  1  1

Output:  126
         142
         143
         311
         331
         332
         312
         341
         351
         352
         440
Example

Node: 126 142 143 311 312 331 332 333 341 351 352 440
Removed?: x x x x x x x x x x x x x
In-degree: 0 0 2 1 2 1 1 2 1 1 1 1
Output: 126 142 143 311 331 332 333 341 351 352 440

CSE 142 → CSE 143 → CSE 311 → CSE 341 → CSE 351 → CSE 333
MATH 126 → CSE 142 → CSE 331 → CSE 332 → CSE 312 → CSE 352 → CSE 440

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A couple of things to note

• Needed a vertex with in-degree of 0 to start
  – No cycles
• Ties between vertices with in-degrees of 0 can be broken arbitrarily
  – Potentially many different correct orders
Topological Sort: Running time?

```java
labelEachVertexWithItsInDegree();
for (ctr=0; ctr < numVertices; ctr++){
    v = findNewVertexOfDegreeZero();
    put v next in output
    for each w adjacent to v
        w.indegree--;
}
```
Topological Sort: Running time?

- What is the worst-case running time?
  - Initialization \( O(|V| + |E|) \) (assuming adjacency list)
  - Sum of all find-new-vertex \( O(|V|^2) \) (because each \( O(|V|) \))
  - Sum of all decrements \( O(|E|) \) (assuming adjacency list)
  - So total is \( O(|V|^2 + |E|) \) – not good for a sparse graph!
**Doing better**

The trick is to avoid searching for a zero-degree node every time!
- Keep the “pending” zero-degree nodes in a list, stack, queue, box, table, or something
- Order we process them affects output but not correctness or efficiency provided add/remove are both $O(1)$

Using a queue:

1. Label each vertex with its in-degree, enqueue 0-degree nodes
2. While queue is not empty
   a) $v = \text{dequeue}()$
   b) Output $v$ and remove it from the graph
   c) For each vertex $u$ adjacent to $v$ (i.e. $u$ such that $(v,u)$ in $E$), decrement the in-degree of $u$, if new degree is 0, enqueue it
Topological Sort (optimized): Running time?

```java
labelAllAndEnqueueZeros();
for (ctr=0; ctr < numVertices; ctr++) {
    v = dequeue();
    put v next in output
    for each w adjacent to v {
        w.indegree--;
        if (w.indegree == 0)
            enqueue(w);
    }
}
```
Topological Sort (optimized): Running time?

```
labelAllAndEnqueueZeros();
for (ctr = 0; ctr < numVertices; ctr++) {
  v = dequeue();
  put v next in output
  for each w adjacent to v {
    w.indegree--;
    if (w.indegree == 0)
      enqueue(w);
  }
}
```

- What is the worst-case running time?
  - Initialization: $O(|V| + |E|)$ (assuming adjacency list)
  - Sum of all enqueues and dequeues: $O(|V|)$
  - Sum of all decrements: $O(|E|)$ (assuming adjacency list)
  - So total is $O(|E| + |V|)$ – much better for sparse graph!
Graph Traversals

Next problem: For an arbitrary graph and a starting node \( v \), find all nodes \( \textit{reachable} \) (i.e., there exists a path) from \( v \)

- Possibly “do something” for each node (an iterator!)
  - E.g. Print to output, set some field, etc.

Related:
- Is an undirected graph connected?
- Is a directed graph weakly / strongly connected?
  - For strongly, need a cycle back to starting node

Basic idea:
- Keep following nodes
- But “mark” nodes after visiting them, so the traversal terminates and processes each reachable node exactly once
traverseGraph(Node start) {
    Set pending = emptySet();
    pending.add(start)
    mark start as visited
    while(pending is not empty) {
        next = pending.remove()
        for each node u adjacent to next
            if(u is not marked) {
                mark u
                pending.add(u)
            }
    }
}
Running time and options

• Assuming add and remove are $O(1)$, entire traversal is $O(|E|)$
  • Use an adjacency list representation

• The order we traverse depends entirely on how add and remove work/are implemented
  – Depth-first graph search (DFS): a stack
  – Breadth-first graph search (BFS): a queue

• DFS and BFS are “big ideas” in computer science
  – Depth: recursively explore one part before going back to the other parts not yet explored
  – Breadth: Explore areas closer to the start node first
Recursive DFS, Example : trees

• A tree is a graph and DFS and BFS are particularly easy to “see”

```
DFS(Node start) {
    mark and “process” (e.g. print) start
    for each node u adjacent to start
        if u is not marked
            DFS(u)
}
```

Order processed: A, B, D, E, C, F, G, H

• Exactly what we called a “pre-order traversal” for trees
• The marking is not needed here, but we need it to support arbitrary graphs, we need a way to process each node exactly once
**DFS with a stack, Example: trees**

DF2(Node start) {
    initialize stack s to hold start
    mark start as visited
    while(s is not empty) {
        next = s.pop() // and “process”
        for each node u adjacent to next
            if(u is not marked)
                mark u and push onto s
    }
}

Order processed:
- A different but perfectly fine traversal
DFS with a stack, Example: trees

DFS2(Node start) {
    initialize stack s to hold start
    mark start as visited
    while(s is not empty) {
        next = s.pop() // and “process”
        for each node u adjacent to next
            if(u is not marked)
                mark u and push onto s
    }
}

Order processed: A, C, F, H, G, B, E, D

• A different but perfectly fine traversal
BFS with a queue, Example: trees

BFS(Node start) {
    initialize queue q to hold start
    mark start as visited
    while(q is not empty) {
        next = q.dequeue() // and “process”
        for each node u adjacent to next
            if(u is not marked)
                mark u and enqueue onto q
    }
}

Order processed:
• A “level-order” traversal
BFS with a queue, Example: trees

BFS(Node start) {
    initialize queue q to hold start
    mark start as visited
    while (q is not empty) {
        next = q.dequeue() // and "process"
        for each node u adjacent to next
            if (u is not marked)
                mark u and enqueue onto q
    }
}

Order processed: A, B, C, D, E, F, G, H
• A “level-order” traversal
DFS/BFS Comparison

Breadth-first search:
• Always finds shortest paths, i.e., “optimal solutions
  – Better for “what is the shortest path from x to y”
• Queue may hold $O(|V|)$ nodes (e.g. at the bottom level of binary tree of height $h$, $2^h$ nodes in queue)

Depth-first search:
• Can use less space in finding a path
  – If longest path in the graph is $p$ and highest out-degree is $d$ then DFS stack never has more than $d*p$ elements

A third approach: Iterative deepening (IDDFS):
  – Try DFS but don’t allow recursion more than $k$ levels deep.
  – If that fails, increment $k$ and start the entire search over
• Like BFS, finds shortest paths. Like DFS, less space.
Saving the path

• Our graph traversals can answer the “reachability question”:
  – “Is there a path from node x to node y?”

• Q: But what if we want to output the actual path?
  – Like getting driving directions rather than just knowing it’s possible to get there!

• A: Like this:
  – Instead of just “marking” a node, store the previous node along the path (when processing u causes us to add v to the search, set v.path field to be u)
  – When you reach the goal, follow path fields backwards to where you started (and then reverse the answer)
  – If just wanted path length, could put the integer distance at each node instead
Example using BFS

What is a path from Seattle to Austin
  – Remember marked nodes are not re-enqueued
  – Note shortest paths may not be unique
Example using BFS

What is a path from Seattle to Austin
  - Remember marked nodes are not re-enqueued
  - Note shortest paths may not be unique